CS 4134 – Quantum Computation and Information Processing

In-class practice problems I

Virginia Tech Department of Computer Science, Spring 2026
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Name:

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Number of problems: 6

Instructions: These problems are for BONUS points that will count towards your grade on Exam 2, if you submit by the end of class. You are not required to complete all problems. The number of bonus points you get will be based on how many problems you complete, and the quality of the solutions. You may work with others to solve the problems, and you may ask the instructor for help as well. There is extra paper available if you need it.

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1. Calculate the following inner products. Explain your reasoning. Throughout, we assume that $\{|k\rangle\}_{k=0}^{d-1}$ is the standard basis, with the dimension $d \in \{2, 3, \dots\}$.
   (a) $\langle 0|3\rangle$
   (b) $\langle 2|2\rangle$
   (c) $\langle v_1|v_2\rangle$, where $|v_1\rangle = (3 + 4\mathrm{i})|0\rangle + (2 - 5\mathrm{i})|1\rangle$ and $|v_2\rangle = (7 + 2\mathrm{i})|0\rangle + (-7 + \mathrm{i})|1\rangle$.

2. Consider the following three-dimensional vector $|v\rangle \in \mathbb{C}^3$, written with respect to the standard basis $\{|0\rangle, |1\rangle, |2\rangle\}$:

$$|v\rangle = \begin{pmatrix} 5 - \mathrm{i} \\ 6 + 3\mathrm{i} \\ 7 - 4\mathrm{i} \end{pmatrix}. \tag{1}$$

(a) Determine the values $\langle 0|v\rangle$, $\langle 1|v\rangle$, and $\langle 2|v\rangle$.

(b) Write the vector $|v\rangle$ in the abstract form $|v\rangle = a|0\rangle + b|1\rangle + c|2\rangle$, with the appropriate values of a, b, and c.

(c) Is the vector normalized? Say why or why not. If not normalized, then normalize it.

3. Consider the vectors

$$|v_1\rangle = \frac{3 + \mathrm{i}\sqrt{3}}{4}|0\rangle + \frac{1}{2}|1\rangle, \tag{2}$$

$$|v_2\rangle = \frac{1}{4}|0\rangle + x|1\rangle. \tag{3}$$

(a) Find the value of x so that $|v_1\rangle$ and $|v_2\rangle$ are orthogonal.
(b) Find the value of x so that $|v_2\rangle$ is normalized.
(c) For what values of x (if any) are $|v_1\rangle$ and $|v_2\rangle$ orthonormal?

4. Let $\{|e_k\rangle\}_{k=1}^d$ and $\{|f_k\rangle\}_{k=1}^d$ be two orthonormal bases for $\mathbb{C}^d$. Prove that $U = \sum_{k=1}^d |e_k\rangle\langle f_k|$ is a unitary operator.

5. (a) State the mathematical definition of the trace of a matrix $M \in \mathrm{L}(\mathbb{C}^d)$.
   (b) For a bipartite matrix M_{AB}, state the definition of the partial traces $\mathrm{Tr}_A[M_{AB}]$ and $\mathrm{Tr}_B[M_{AB}]$.
   (c) Calculate the partial traces $\mathrm{Tr}_A[M_{AB}]$ and $\mathrm{Tr}_B[M_{AB}]$ for the following two-qubit linear operators. Both A and B are qubits.
      i. $M_{AB} = \begin{pmatrix} 1-\mathrm{i} & 2+3\mathrm{i} & -5 & \mathrm{i} \\ -4 & 7-2\mathrm{i} & 1+2\mathrm{i} & 0 \\ 1 & 4\mathrm{i} & -2 & 6+3\mathrm{i} \\ 2\mathrm{i} & 1 & -\mathrm{i} & 3-\mathrm{i} \end{pmatrix}$
      ii. $M_{AB} = \begin{pmatrix} 3 & 1-\mathrm{i} & 0 & \mathrm{i} \\ 1+\mathrm{i} & 2 & 4\mathrm{i} & 0 \\ 0 & -4\mathrm{i} & 1 & -9+2\mathrm{i} \\ -3\mathrm{i} & 0 & -9-2\mathrm{i} & 9 \end{pmatrix}$
   (d) Consider the three-qubit state vectors

$$|\psi_1\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0,0,0\rangle_{ABC} + |1,1,1\rangle_{ABC}), \tag{4}$$

$$|\psi_2\rangle_{ABC} = \frac{1}{\sqrt{3}}(|0,0,1\rangle_{ABC} + |0,1,0\rangle_{ABC} + |1,0,0\rangle_{ABC}). \tag{5}$$

   Are either entangled states? Briefly justify your answer.
   (e) Determine the following partial traces.
      i. $\mathrm{Tr}_A\left[|\psi_1\rangle\langle\psi_1|_{ABC}\right]$.
      ii. $\mathrm{Tr}_B\left[|\psi_2\rangle\langle\psi_2|_{ABC}\right]$.
   (f) Is there any entanglement present in the states after taking the partial trace?
   (g) Which state, $|\psi_1\rangle_{ABC}$ or $|\psi_2\rangle_{ABC}$, is more resilient to qubit loss? Explain why.

6. Consider the bipartite quantum state $|\Psi^-\rangle\langle\Psi^-|_{AB}$ of Alice and Bob. In this problem, we show that Alice and Bob's measurement outcomes are always anti-correlated when they measure in the same basis, regardless of the chosen basis.

   (a) If Alice and Bob both measure in the $\{|0\rangle, |1\rangle\}$ basis, what is the distribution of outcomes? Justify that their outcomes are *anti-correlated*: whenever Alice gets $0/1$, Bob gets $1/0$.

   (b) If instead Alice and Bob measure in the $\{|+\rangle, |-\rangle\}$ basis, then show that their outcomes are still anti-correlated: whenever Alice gets $+/-$, Bob gets $-/+$.

   (c) Let U be an arbitrary 2×2 unitary matrix. Prove that $(U \otimes U)|\Psi^-\rangle\langle\Psi^-|(U \otimes U)^\dagger = |\Psi^-\rangle\langle\Psi^-|$.

   (d) Let U be an arbitrary 2×2 unitary matrix. Prove that the vectors $U^\dagger|0\rangle$ and $U^\dagger|1\rangle$ define a measurement.

   (e) Using the result from (c), prove that if Alice and Bob both measure with respect to $\{U^\dagger|0\rangle, U^\dagger|1\rangle\}$, with U being an arbitrary 2×2 unitary matrix, then their measurement outcomes are anti-correlated. Thus, regardless of the basis choice, their measurement outcomes are anti-correlated.