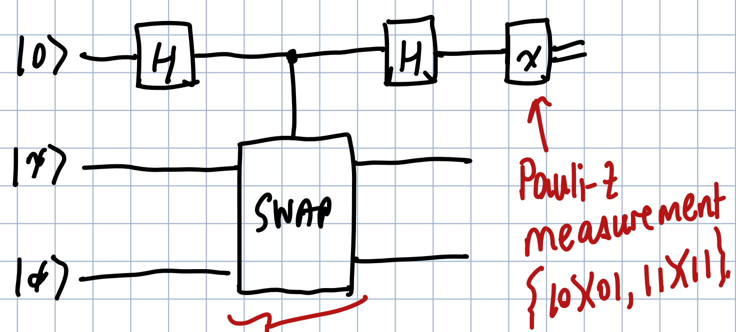
① Recap : SWAP test

- **Goal:** given two pure states given by $|\psi\rangle$ and $|\phi\rangle$, find out how close they are.



$$\Pr[0] = \frac{1}{2}\left(1 + |\langle\psi|\phi\rangle|^2\right)$$

$$\Pr[1] = \frac{1}{2}\left(1 - |\langle\psi|\phi\rangle|^2\right)$$

$H|0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) = |+\rangle$

$H|1\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right) = |-\rangle$

$$\left.\begin{aligned} |0\rangle|0\rangle &\mapsto |0\rangle|0\rangle \\ |0\rangle|1\rangle &\mapsto |1\rangle|0\rangle \\ |1\rangle|0\rangle &\mapsto |0\rangle|1\rangle \\ |1\rangle|1\rangle &\mapsto |1\rangle|1\rangle \end{aligned}\right\} \rightarrow \text{This means that } \mathrm{SWAP}\left(|\psi\rangle|\phi\rangle\right) = |\phi\rangle|\psi\rangle$$

- The quantity $|\langle\psi|\phi\rangle|^2$ quantifies the closeness of $|\psi\rangle$ and $|\phi\rangle$ — it is called the fidelity of $|\psi\rangle$ and $|\phi\rangle$.

$\{M_x\}_x$, $M_x \geq 0$, $\sum_x M_x = \mathbb{1}$.

(*) Observe that $\{|\phi\rangle\langle\phi|, \mathbb{1} - |\phi\rangle\langle\phi|\}$ is a measurement (POVM).

Check: $\langle v|\phi\rangle\langle\phi|v\rangle = |\langle v|\phi\rangle|^2 \geq 0$ ✓

Also, from the Cauchy-Schwarz inequality: $|\langle v|\phi\rangle|^2 \leq \langle v|v\rangle \cdot \underbrace{\langle\phi|\phi\rangle}_{=1}$

$= \langle v|v\rangle = \||v\rangle\|^2$

Therefore, $\langle v|\left(\mathbb{1} - |\phi\rangle\langle\phi|\right)|v\rangle = \langle v|v\rangle - \underbrace{|\langle v|\phi\rangle|^2}_{\leq \langle v|v\rangle} \geq 0$ ✓

Finally, $|\phi\rangle\langle\phi| + \mathbb{1} - |\phi\rangle\langle\phi| = \mathbb{1}$ ✓

So $\{|\phi\rangle\langle\phi|, \mathbb{1} - |\phi\rangle\langle\phi|\}$ satisfies the definition of a quantum measurement (POVM).

(*) $\{|\phi\rangle\langle\phi|, \mathbb{1} - |\phi\rangle\langle\phi|\}$ is a measurement that tells us whether or not a given state is equal to $|\phi\rangle\langle\phi|$.

It has two outcomes (b/c the set has two operators).

- $|\phi\rangle\langle\phi| \equiv$ "yes, the state is $|\phi\rangle\langle\phi|$"
- $\mathbb{1} - |\phi\rangle\langle\phi| \equiv$ "no, the state is not $|\phi\rangle\langle\phi|$".

Born Rule: $\{M_x\}$ POVM
State $\rho \rightarrow \Pr(x) = \mathrm{Tr}[M_x \rho]$.

(*) For a given state ρ, $\Pr[\phi] = \mathrm{Tr}[|\phi\rangle\langle\phi|\rho]$

$$\Pr[\text{not } \phi] = \mathrm{Tr}[(\mathbb{1} - |\phi\rangle\langle\phi|)\rho] = \mathrm{Tr}(\rho) - \mathrm{Tr}[|\phi\rangle\langle\phi|\rho] = 1 - \Pr[\phi].$$

- If $\rho = |\phi\rangle\langle\phi|$ itself, then $\Pr(\phi) = \mathrm{Tr}[|\phi\rangle\underbrace{\langle\phi|\phi\rangle}_{=1}\langle\phi|] = \mathrm{Tr}[|\phi\rangle\langle\phi|] = \langle\phi|\phi\rangle = 1.$ ✓

$$\Pr(\text{not } \phi) = 1 - \Pr(\phi) = 0$$

- If $\rho = |\psi\rangle\langle\psi|$, then $\Pr(\phi) = \mathrm{Tr}[|\phi\rangle\langle\phi|\psi\rangle\langle\psi|] = |\langle\phi|\psi\rangle|^2 \hookrightarrow$ fidelity!

(*) So the swap test allows us to do the measurement $\{|\phi\rangle\langle\phi|, \mathbb{1} - |\phi\rangle\langle\phi|\}$ and estimate the fidelity — without even knowing what $|\phi\rangle$ is!

② Statistical Estimation from a Quantum Computer.

- When we run a quantum algorithm and do the measurement at the end, the outcomes will generally be probabilistic.
- To extract the relevant information, we have to run the algorithm many times.

(a) From the swap test, we know that $\Pr[0] = \frac{1}{2}(1+\alpha)$ $\alpha = |\langle\psi|\phi\rangle|^2$.
$\Pr[1] = \frac{1}{2}(1-\alpha)$

Running this many times will give us a bunch of "0"s and "1"s.

How do we use the "0"s and "1"s to estimate α?

(b) Procedure: For $i = 1, 2, \dots, N$ ($N \equiv$ number of samples)

- Each time we get outcome "0" → record $x_i = 1$
- Each time we get outcome "1" → record $x_i = -1$
- Do this N times, then take the sample mean/average: $\hat{x}_N = \frac{1}{N}\sum_{i=1}^{N} x_i$

⊛ This defines a random variable X:

$\Pr[X = \pm 1] = \frac{1}{2}(1 \pm \alpha)$

This defines a random variable $\hat{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$. $\hat{X}_N$ is an unbiased estimator of X:

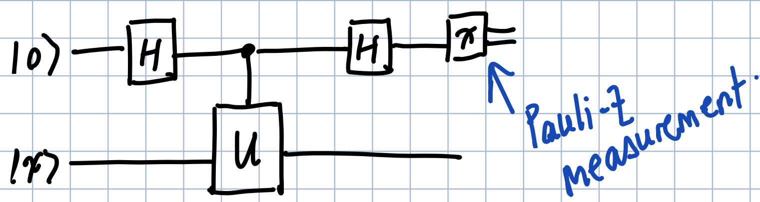
$$\mathbb{E}[\hat{X}_N] = \frac{1}{N}\sum_{i=1}^{N} \underbrace{\mathbb{E}[X_i]}_{=\mathbb{E}[X]\ \forall i,\ \text{b/c all samples are independent and identical.}} = \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}[X] = \mathbb{E}[X].$$

$$\mathbb{E}[X] = (+1)\cdot\Pr[X=+1] + (-1)\cdot\Pr[X=-1] = \frac{1}{2}(1+\alpha) - \frac{1}{2}(1-\alpha) = \alpha.$$

- As $N \to \infty$, $\hat{X}_N \to \mathbb{E}[X] = |\langle\psi|\phi\rangle|^2$ (law of large numbers).

⊛ So the sample average approaches the true (unknown) value of α!

③ **Hadamard Test**: Estimating $\langle\psi|U|\psi\rangle$, where U is a unitary and $|\psi\rangle$ is a state vector.



$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

$$|\psi_{init}\rangle = |0\rangle|\psi\rangle \mapsto |+\rangle|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle\right) \mapsto \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle\right)$$

$$\mapsto \frac{1}{\sqrt{2}}\left(|+\rangle|\psi\rangle + |-\rangle U|\psi\rangle\right) = \frac{1}{\sqrt{2}}\left(|0\rangle\frac{1}{\sqrt{2}}(|\psi\rangle + U|\psi\rangle) + |1\rangle\frac{1}{\sqrt{2}}(|\psi\rangle - U|\psi\rangle)\right) = |\psi_{final}\rangle.$$

$$\Pr[0] = \mathrm{Tr}\left[(|0\rangle\langle 0| \otimes \mathbb{1})|\psi_{final}\rangle\langle\psi_{final}|\right]$$

$\frac{1}{2}\left(|\psi\rangle + U|\psi\rangle\right)^\dagger = \frac{1}{2}\left(\langle\psi| + \langle\psi|U^\dagger\right)$

$$= \frac{1}{4}\left(\left(\langle\psi| + \langle\psi|U^\dagger\right)\left(|\psi\rangle + U|\psi\rangle\right)\right)$$

$$= \frac{1}{4}\left(\langle\psi|\psi\rangle + \langle\psi|U|\psi\rangle + \underbrace{\langle\psi|U^\dagger|\psi\rangle}_{=\overline{\langle\psi|U|\psi\rangle}} + \langle\psi|\underbrace{U^\dagger U}_{=\mathbb{1}}|\psi\rangle\right)$$

$\hookrightarrow = \langle\psi|\psi\rangle = 1$

$z = x + yi$, $\bar{z} = x - yi$ $\Big\} \to z + \bar{z} = 2x$

$x = \mathrm{Re}(z).$

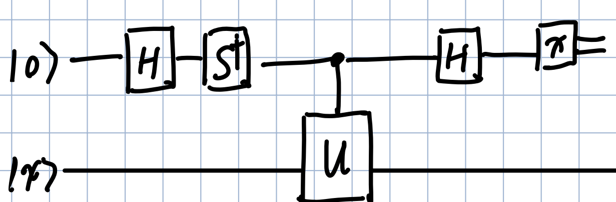
$$= \frac{1}{2}\left(1 + \underbrace{\mathrm{Re}(\langle\psi|U|\psi\rangle)}_{\alpha}\right)$$

$$\Pr[1] = \frac{1}{2}\left(1 - \underbrace{\mathrm{Re}(\langle\psi|U|\psi\rangle)}_{\alpha}\right)$$

→ Then same procedure as above to estimate α!

$\Pr[0] = \frac{1}{2}(1+\alpha)$, $\Pr[1] = \frac{1}{2}(1-\alpha)$.

⊛ For the imaginary part → include the S gate!



$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \Rightarrow S^\dagger = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$

$S^\dagger|0\rangle = |0\rangle$

$S^\dagger|1\rangle = -i|1\rangle$

$$|0\rangle|\psi\rangle \mapsto |+\rangle|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle\right) \mapsto \frac{1}{\sqrt{2}}\left(\overbrace{|0\rangle}^{S^\dagger|0\rangle}|\psi\rangle \overbrace{-i|1\rangle}^{S^\dagger|1\rangle}|\psi\rangle\right)$$

$$\mapsto \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle - i|1\rangle U|\psi\rangle\right) \mapsto \frac{1}{\sqrt{2}}\left(|+\rangle|\psi\rangle - i|-\rangle U|\psi\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|\gamma\rangle - i\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)U|\gamma\rangle\right)$$

$$= \frac{1}{2}\left(|0\rangle|\gamma\rangle + |1\rangle|\gamma\rangle - i|0\rangle U|\gamma\rangle + i|1\rangle U|\gamma\rangle\right)$$

$$= \frac{1}{2}\left(|0\rangle(|\gamma\rangle - iU|\gamma\rangle) + |1\rangle(|\gamma\rangle + iU|\gamma\rangle)\right) \rightarrow (|\gamma\rangle - iU|\gamma\rangle)^\dagger = \langle\gamma| + i\langle\gamma|U^\dagger$$

$$\Rightarrow \Pr[0] = \frac{1}{4}\left(\langle\gamma| + i\langle\gamma|U^\dagger\right)\left(|\gamma\rangle - iU|\gamma\rangle\right) = \frac{1}{4}\Big(\underbrace{\langle\gamma|\gamma\rangle}_{=1} - i\langle\gamma|U|\gamma\rangle + i\langle\gamma|U^\dagger|\gamma\rangle$$

$$+ \underbrace{\langle\gamma|\underbrace{U^\dagger U}_{=\mathbb{1}}|\gamma\rangle}_{=1}\Big)$$

$$= \frac{1}{4}\Big(1 - i\overbrace{\langle\gamma|U|\gamma\rangle}^{z} + i\overbrace{\overline{\langle\gamma|U|\gamma\rangle}}^{\bar{z}} + 1\Big) \qquad (z - \bar{z} = x+iy-x+iy = 2iy)$$

$$= \frac{1}{4}\Big(2 - i\underbrace{\left(\langle\gamma|U|\gamma\rangle - \overline{\langle\gamma|U|\gamma\rangle}\right)}_{=2i\,\mathrm{Im}(\langle\gamma|U|\gamma\rangle)}\Big)$$

$$= \frac{1}{2}\Big(1 + \underbrace{\mathrm{Im}(\langle\gamma|U|\gamma\rangle)}_{\alpha}\Big).$$

$$\Pr[1] = \frac{1}{2}\Big(1 - \underbrace{\mathrm{Im}(\langle\gamma|U|\gamma\rangle)}_{\alpha}\Big).$$

→ Then same procedure as above to estimate α!

$\Pr[0] = \frac{1}{2}(1+\alpha)$, $\Pr[1] = \frac{1}{2}(1-\alpha)$.