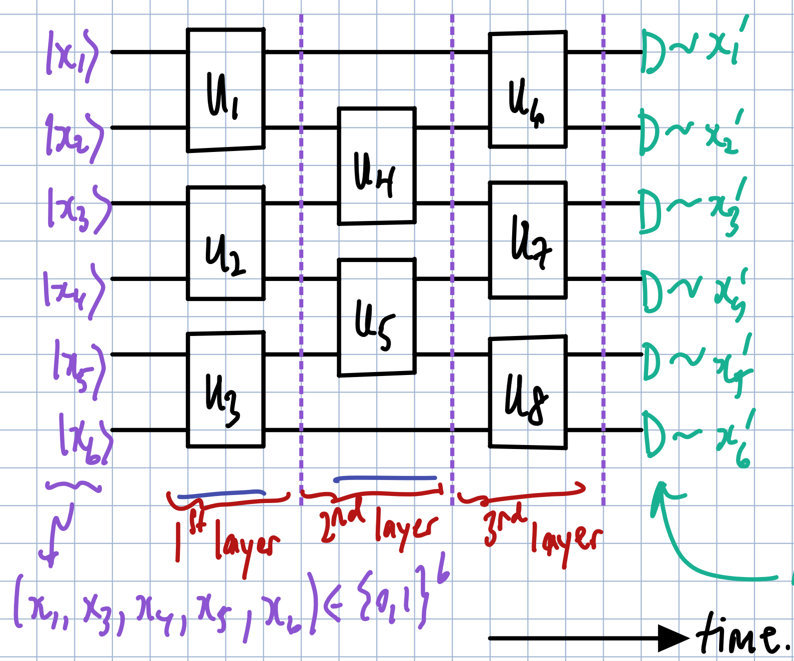
① Recap: Quantum Circuits



(*) The circuit elements/gates are **unitaries**.

$(U: U^\dagger U = U U^\dagger = \mathbb{1})$

Measurement/read-out

$\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ → - Pauli-Z
- Computational Basis.

(a) Pauli gates:

$-\boxed{X}- \rightarrow X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (X|0\rangle = |1\rangle,\ X|1\rangle = |0\rangle)$

$-\boxed{Y}- \rightarrow Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (Y|0\rangle = i|1\rangle,\ Y|1\rangle = -i|0\rangle)$

$-\boxed{Z}- \rightarrow Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (Z|0\rangle = |0\rangle,\ Z|1\rangle = -|1\rangle)$

(b) Hadamard gate:

$-\boxed{H}- \rightarrow H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$

(c) <u>Phase gate</u> :

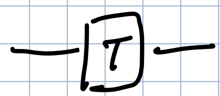
$\rightarrow S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$ $\quad S|0\rangle = |0\rangle$

$S|1\rangle = i|1\rangle$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$S|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = |0\rangle$

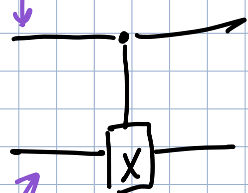
(d) <u>T - gate</u> :



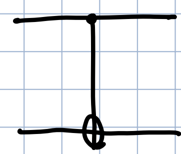
$\rightarrow T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

(e) <u>CNOT / CX gate</u> :

"controlled-X"



or



$\circledast$ $|a,b\rangle \mapsto |a, a\oplus b\rangle$

$$\text{CNOT} = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array}\begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \end{array}$$

$|0\rangle|0\rangle \longmapsto |0\rangle|0\rangle$

$|0\rangle|1\rangle \longmapsto |0\rangle|1\rangle$

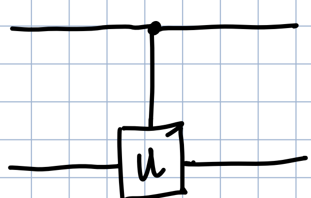
$|1\rangle|0\rangle \longmapsto |1\rangle X|0\rangle = |1\rangle|1\rangle$

$|1\rangle|1\rangle \longmapsto |1\rangle X|1\rangle = |1\rangle|0\rangle$

B/c of linearity, this determines the action on <u>any</u> state!

$|\psi\rangle = a|0,0\rangle + b|0,1\rangle + c|1,0\rangle + d|1,1\rangle \mapsto a|0,0\rangle + b|0,1\rangle + c|1,1\rangle + d|1,0\rangle$

(f) <u>Controlled Unitary</u>



$|0\rangle|0\rangle \longmapsto |0\rangle|0\rangle$

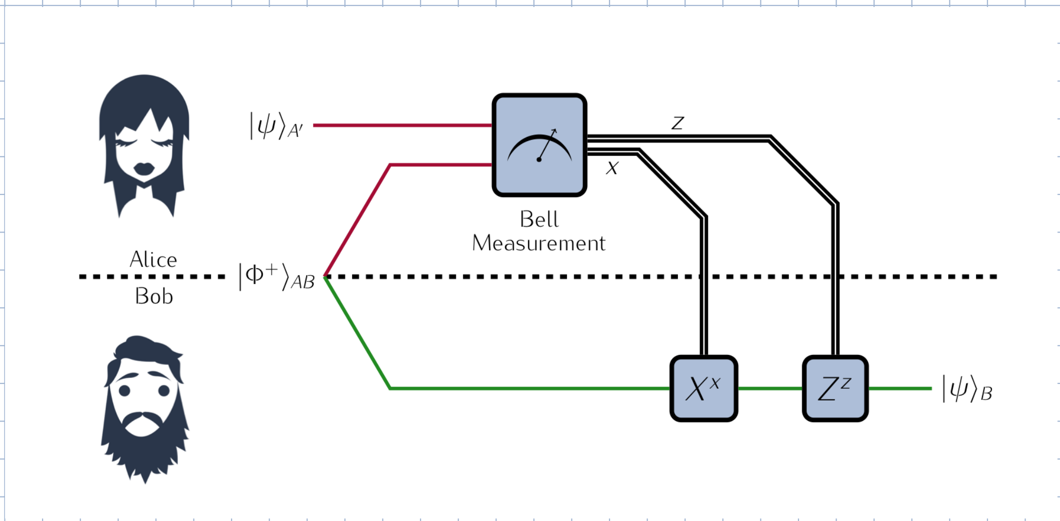
$|0\rangle|1\rangle \longmapsto |0\rangle|1\rangle$

$|1\rangle|0\rangle \longmapsto |1\rangle U|0\rangle$

$|1\rangle|1\rangle \longmapsto |1\rangle U|1\rangle$

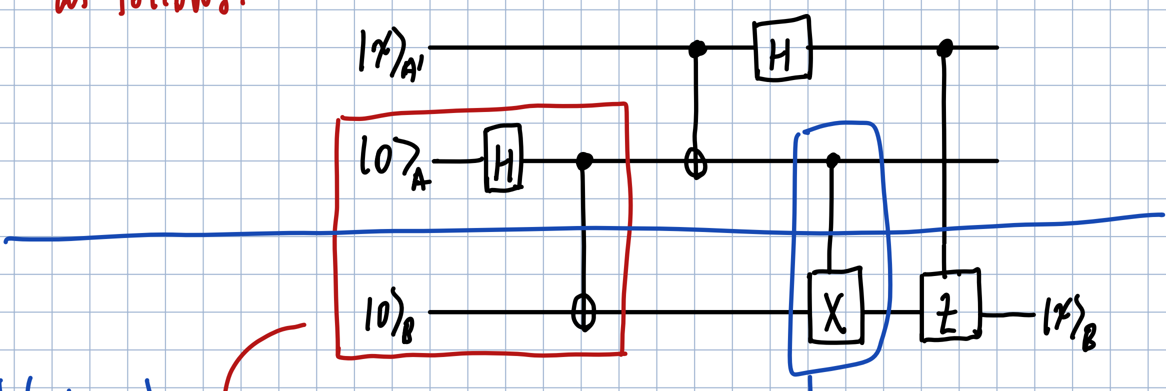
$$cU = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U.$$

• Example: Teleportation



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$

⊛ If Alice and Bob are not spatially separated, then the algorithm can be modified as follows:



CNOT

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$|0\rangle_A|0\rangle_B \mapsto |+\rangle_A|0\rangle_B = \tfrac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|0\rangle_B) \mapsto \tfrac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) = |\Phi^+\rangle$$

$$|\psi\rangle_{A'}|0\rangle_A|0\rangle_B \longmapsto |\psi\rangle_{A'} \otimes |\Phi^+\rangle_{AB} = \tfrac{1}{\sqrt{2}}(|\psi\rangle_{A'}|0\rangle_A|0\rangle_B + |\psi\rangle_{A'}|1\rangle_A|1\rangle_B)$$

$|\psi\rangle_{A'} = \alpha|0\rangle + \beta|1\rangle$

$$= \tfrac{1}{\sqrt{2}}\big(\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \beta|1\rangle_{A'}|0\rangle_A|0\rangle_B + \alpha|0\rangle_{A'}|1\rangle_A|1\rangle_B + \beta|1\rangle_{A'}|1\rangle_A|1\rangle_B\big).$$

$$\xmapsto{CNOT_{A'A}} \tfrac{1}{\sqrt{2}}\big(\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \beta|1\rangle_{A'}|1\rangle_A|0\rangle_B + \alpha|0\rangle_{A'}|1\rangle_A|1\rangle_B + \beta|1\rangle_{A'}|0\rangle_A|1\rangle_B\big).$$

$$\overset{}{\underset{H_{A'}}{\longmapsto}} \frac{1}{2}\Big(\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|0\rangle_A|0\rangle_B$$
$$+ \beta|0\rangle_{A'}|1\rangle_A|0\rangle_B - \beta|1\rangle_{A'}|1\rangle_A|0\rangle_B$$
$$+ \alpha|0\rangle_{A'}|1\rangle_A|1\rangle_B + \alpha|1\rangle_{A'}|1\rangle_A|1\rangle_B$$
$$+ \beta|0\rangle_{A'}|0\rangle_A|1\rangle_B - \beta|1\rangle_{A'}|0\rangle_A|1\rangle_B\Big).$$

$$\underset{CNOT_{AB}}{\longmapsto} \frac{1}{2}\Big(\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|0\rangle_A|0\rangle_B$$
$$+ \beta|0\rangle_{A'}|1\rangle_A|1\rangle_B - \beta|1\rangle_{A'}|1\rangle_A|1\rangle_B$$
$$+ \alpha|0\rangle_{A'}|1\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|1\rangle_A|0\rangle_B \qquad Z|1\rangle = -|1\rangle$$
$$+ \beta|0\rangle_{A'}|0\rangle_A|1\rangle_B - \beta|1\rangle_{A'}|0\rangle_A|1\rangle_B\Big). \qquad Z|0\rangle = |0\rangle$$

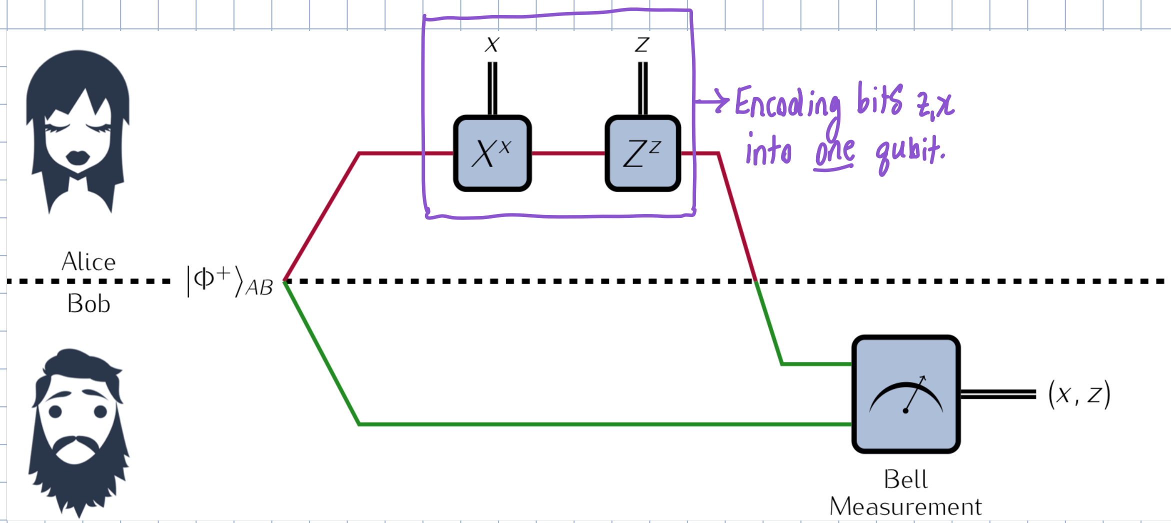
$$\underset{CZ_{A'B}}{\longmapsto} \frac{1}{2}\Big(\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|0\rangle_A|0\rangle_B$$
$$+ \beta|0\rangle_{A'}|1\rangle_A|1\rangle_B + \beta|1\rangle_{A'}|1\rangle_A|1\rangle_B$$
$$+ \alpha|0\rangle_{A'}|1\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|1\rangle_A|0\rangle_B$$
$$+ \beta|0\rangle_{A'}|0\rangle_A|1\rangle_B + \beta|1\rangle_{A'}|0\rangle_A|1\rangle_B\Big).$$

$$= \frac{1}{2}\Big(|0\rangle_{A'}|0\rangle_A \overbrace{\left(\alpha|0\rangle_B + \beta|1\rangle_B\right)}^{|\psi\rangle}$$
$$+ |0\rangle_{A'}|1\rangle_A \underbrace{\left(\alpha|0\rangle_B + \beta|1\rangle_B\right)}_{|\psi\rangle}$$
$$+ |1\rangle_{A'}|0\rangle_A \left(\alpha|0\rangle_B + \beta|1\rangle_B\right)$$
$$+ |1\rangle_{A'}|1\rangle_A \left(\alpha|0\rangle_B + \beta|1\rangle_B\right)\Big)$$

$$= \frac{1}{2}\left(|0\rangle_{A'}|0\rangle_A + |0\rangle_{A'}|1\rangle_A + |1\rangle_{A'}|0\rangle_A + |1\rangle_{A'}|1\rangle_A\right)|\psi\rangle_B$$

$$= \underline{\underline{|+\rangle_{A'}|+\rangle_A|\psi\rangle_B}}$$

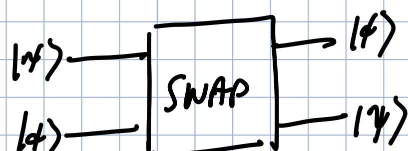
- __Example__: Superdense coding: using entanglement + 1 qubit to transmit 2 classical bits.



② The Swap test

✱ How do we estimate the inner product of two (unknown) states on a quantum computer?

- The swap gate:



$|0\rangle|0\rangle \mapsto |0\rangle|0\rangle$

$|0\rangle|1\rangle \mapsto |1\rangle|0\rangle$

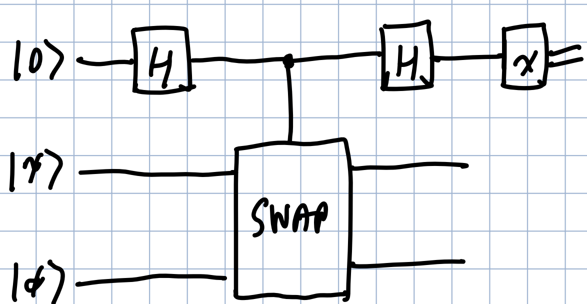
$|1\rangle|0\rangle \mapsto |0\rangle|1\rangle$

$|1\rangle|1\rangle \mapsto |1\rangle|1\rangle$

As a matrix:

$$\begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

- Circuit for the SWAP test



$$|\psi_{init}\rangle = |0\rangle|\gamma\rangle|\phi\rangle \overset{H}{\mapsto} \frac{1}{\sqrt{2}}\left(|0\rangle|\gamma\rangle|\phi\rangle + |1\rangle|\gamma\rangle|\phi\rangle\right)$$

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\overset{cSWAP}{\mapsto} \frac{1}{\sqrt{2}}\left(|0\rangle|\gamma\rangle|\phi\rangle + |1\rangle|\phi\rangle|\gamma\rangle\right)$$

$$\overset{H}{\mapsto} \frac{1}{\sqrt{2}}\left(|+\rangle|\gamma\rangle|\phi\rangle + |-\rangle|\phi\rangle|\gamma\rangle\right) \qquad |\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle).$$

$\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$.

$$|\psi_{final}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle\frac{1}{\sqrt{2}}\left(|\gamma\rangle|\phi\rangle + |\phi\rangle|\gamma\rangle\right) + |1\rangle\frac{1}{\sqrt{2}}\left(|\gamma\rangle|\phi\rangle - |\phi\rangle|\gamma\rangle\right)\right)$$
$$= \frac{1}{2}|0\rangle\left(|\gamma\rangle|\phi\rangle + |\phi\rangle|\gamma\rangle\right) + \frac{1}{2}|1\rangle\left(|\gamma\rangle|\phi\rangle - |\phi\rangle|\gamma\rangle\right)$$

Now measure — what is the probability of getting zero? (Recall partial measurements).

$$Pr[0] = Tr\left[\left(|0\rangle\langle 0| \otimes \mathbb{1} \otimes \mathbb{1}\right)|\psi_{final}\rangle\langle\psi_{final}|\right].$$

$$= \underbrace{\frac{1}{2}\left(|\gamma\rangle|\phi\rangle + |\phi\rangle|\gamma\rangle\right)}_{|v\rangle}.$$

$$= \frac{1}{4} Tr\left[\left(|\gamma\rangle|\phi\rangle + |\phi\rangle|\gamma\rangle\right)\left(\langle\gamma|\langle\phi| + \langle\phi|\langle\gamma|\right)\right] \qquad Tr\left[|v\rangle\langle v|\right] = \langle v|v\rangle$$

$$= \frac{1}{4}\left(\langle\gamma|\langle\phi| + \langle\phi|\langle\gamma|\right)\left(|\gamma\rangle|\phi\rangle + |\phi\rangle|\gamma\rangle\right)$$

$$= \frac{1}{4}\left(\underbrace{\langle\gamma|\gamma\rangle\langle\phi|\phi\rangle}_{=1} + \underbrace{\langle\gamma|\phi\rangle\langle\phi|\gamma\rangle + \langle\phi|\gamma\rangle\langle\gamma|\phi\rangle}_{=2|\langle\gamma|\phi\rangle|^2} + \underbrace{\langle\phi|\phi\rangle\langle\gamma|\gamma\rangle}_{=1}\right)$$

$$Pr[0] = \frac{1}{2}\left(1 + |\langle\gamma|\phi\rangle|^2\right)$$

$$Pr[1] = \frac{1}{2}\left(1 - |\langle\gamma|\phi\rangle|^2\right)$$

⊛ The probabilities contain the inner product!
So how do we extract the value of the inner product?
We run the algorithm many times!

- Each time we get outcome "0" → record $x_i = 1$
- Each time we get outcome "1" → record $x_i = -1$

⊛ This defines a random variable X:

$$Pr[X = \pm 1] = \frac{1}{2}\left(1 \pm |\langle\gamma|\phi\rangle|^2\right).$$

- Do this N times, then take the sample mean/average: $\hat{x}_N = \frac{1}{N}\sum_{i=1}^{N} x_i$

This defines a random variable $\hat{X}_N = \frac{1}{N}\sum_{i=1}^{N} X_i$. $\hat{X}_N$ is an unbiased estimator of X:

$$\mathbb{E}[\hat{X}_N] = \frac{1}{N}\sum_{i=1}^{N} \underbrace{\mathbb{E}[X_i]}_{=\mathbb{E}[X]\ \forall i,\ \text{b/c all samples are independent and identical.}} = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}[X] = \mathbb{E}[X].$$

$$\mathbb{E}[X] = \sum_{x} x \Pr[X=x]$$

$$\mathbb{E}[X] = (+1)\cdot\Pr[X=+1] + (-1)\cdot\Pr[X=-1] = \tfrac{1}{2}\left(1+|\langle\psi|\phi\rangle|^2\right) - \tfrac{1}{2}\left(1-|\langle\psi|\phi\rangle|^2\right) = |\langle\psi|\phi\rangle|^2$$

- As $N\to\infty$, $\hat{X}_N \to \mathbb{E}[X] = |\langle\psi|\phi\rangle|^2$ (law of large numbers).

(*) So the sample average approaches the true (unknown) inner product!

[89]:
```python
import numpy as np
import matplotlib.pyplot as plt

inner_product=0.8
# Parameters
p = (1/2)*(1-inner_product)            # probability of original "1"
n_samples = 20000

# Step 1: sample Bernoulli (0 or 1)
samples = np.random.binomial(1, p, size=n_samples)

# Step 2: map to +1 / -1
mapped = 1 - 2 * samples    # 0 -> 1, 1 -> -1

# Step 3: running average
running_avg = np.cumsum(mapped) / np.arange(1, n_samples + 1)

# True mean of the new variable
true_mean = 1 - 2*p

# Plot
plt.figure()
plt.plot(running_avg)#, label="Running average")
plt.axhline(true_mean, linestyle='--', label="True inner product")

plt.xlabel("Number of samples")
plt.ylabel("Sample average")
plt.title("Inner product estimation (swap test)")
plt.legend()
plt.show()
```