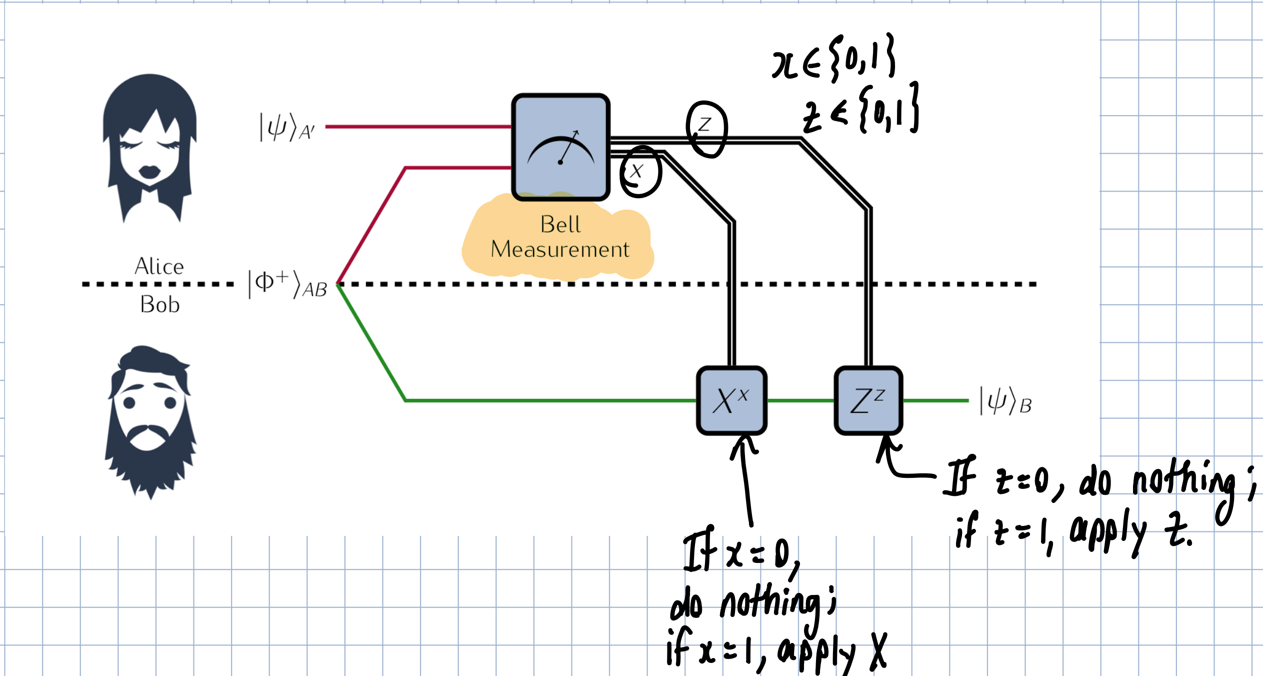
① Recap: Quantum teleportation

- A method to send arbitrary quantum information using entanglement and two bits of classical information.



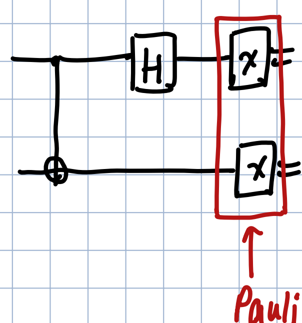
⊛ Teleportation is not instantaneous / faster than light — it only works if Bob gets Alice's measurement outcomes → this takes time!

- The Bell measurement is a critical component.

It is given by the POVM $\{\Phi^+, \Phi^-, \Psi^+, \Psi^-\}$, $\Phi^{\pm} = |\Phi^{\pm}\rangle\langle\Phi^{\pm}|$, $\Psi^{\pm} = |\Psi^{\pm}\rangle\langle\Psi^{\pm}|$,

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle \pm |1\rangle|1\rangle\right), \quad |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle \pm |1\rangle|0\rangle\right).$$

Circuit implementation:



$|0\rangle|0\rangle \mapsto |0\rangle|0\rangle$

$|0\rangle|1\rangle \mapsto |0\rangle|1\rangle$

$|1\rangle|0\rangle \mapsto |1\rangle|1\rangle$

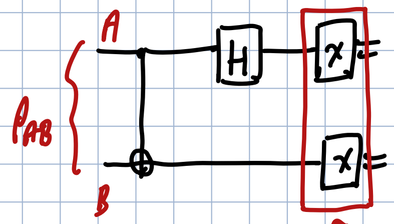
$|1\rangle|1\rangle \mapsto |1\rangle|0\rangle$

Proof: Let ρ_{AB} be the initial state

Let $U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ be the CNOT gate, and $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ the Hadamard gate.

$H|0\rangle = |+\rangle$

$H|1\rangle = |-\rangle \quad |\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$



Pauli-Z measurement.

$$\rho_{AB} \mapsto U\rho_{AB}U^\dagger \mapsto \underbrace{(H\otimes \mathbb{1})U\rho_{AB}U^\dagger(H^\dagger\otimes\mathbb{1})}_{\text{State just before measurement.}}$$

Now do a Pauli-Z measurement on each qubit.
The POVM is $\{|0,0\rangle\langle 0,0|, |0,1\rangle\langle 0,1|, |1,0\rangle\langle 1,0|, |1,1\rangle\langle 1,1|\}$.

Outcome probabilities:

$$\Pr[z,x] = \mathrm{Tr}\Big[|z,x\rangle\langle z,x|_{AB}\,\underbrace{(H\otimes\mathbb{1})U\rho_{AB}U^\dagger(H^\dagger\otimes\mathbb{1})}_{\mathrm{Tr}[|z,x\rangle\langle z,x|\,\sigma]}\Big] \qquad \mathrm{Tr}[ABC]=\mathrm{Tr}[CAB]=\mathrm{Tr}[BCA]$$

$$= \mathrm{Tr}\big[(H^\dagger\otimes\mathbb{1})|z,x\rangle\langle z,x|_{AB}(H\otimes\mathbb{1})U\rho_{AB}U^\dagger\big] \quad \text{(cyclicity of trace)}$$

$$= \mathrm{Tr}\big[\underbrace{U^\dagger(H^\dagger\otimes\mathbb{1})|z,x\rangle\langle z,x|_{AB}(H\otimes\mathbb{1})U}_{M_{z,x}}\,\rho_{AB}\big] \quad \text{(cyclicity of trace again).}$$

$U^\dagger = U$ (b/c CNOT is Hermitian)
$H^\dagger = H$ (b/c Hadamard is also Hermitian). $\Rightarrow M_{z,x} = U(H\otimes\mathbb{1})|z,x\rangle\langle z,x|(H\otimes\mathbb{1})U$

(1) $U(H\otimes\mathbb{1})|0,0\rangle = \mathrm{CNOT}|+,0\rangle = \mathrm{CNOT}\frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|0\rangle)$

$$= \frac{1}{\sqrt{2}}(\mathrm{CNOT}|0\rangle|0\rangle + \underbrace{\mathrm{CNOT}|1\rangle|0\rangle}_{=|1\rangle|1\rangle})$$

$$= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$

$$= |\Phi^+\rangle$$

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\Rightarrow M_{0,0} = \Phi^+$

(2) $U(H\otimes\mathbb{1})|0,1\rangle = \mathrm{CNOT}|+,1\rangle = \mathrm{CNOT}\frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|1\rangle)$

$$= \frac{1}{\sqrt{2}}(\mathrm{CNOT}|0\rangle|1\rangle + \mathrm{CNOT}|1\rangle|1\rangle)$$

$$\downarrow$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|0\rangle\right)$$

$$= |\Psi^+\rangle$$

$\Rightarrow M_{0,1} = \Psi^+$

(3) $U(H\otimes \mathbb{1})|1,0\rangle = CNOT|-,0\rangle = CNOT\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |1\rangle|0\rangle\right)$

$$\downarrow$$

$$= \frac{1}{\sqrt{2}}\left(CNOT|0\rangle|0\rangle - CNOT|1\rangle|0\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |1\rangle|1\rangle\right)$$

$$= |\Phi^-\rangle$$

$\Rightarrow M_{1,0} = \Phi^-$

(4) $U(H\otimes \mathbb{1})|1,1\rangle = CNOT|-,1\rangle = CNOT\frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|1\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(CNOT|0\rangle|1\rangle - CNOT|1\rangle|1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right)$$

$$= |\Psi^-\rangle$$

$\Rightarrow M_{1,1} = \Psi^-$. $\blacksquare$

Also, $\Phi^+ + \Phi^- + \Psi^+ + \Psi^- = \mathbb{1}$.

Proof: Go to the matrix representation.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right) = \begin{array}{cc} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} & \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \end{array}$$

$$\Rightarrow \Phi^+ = |\Phi^+\rangle\langle\Phi^+| = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle - |1\rangle|1\rangle\right) = \begin{array}{c} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \end{array} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}$$

$$\Rightarrow \Phi^- = |\Phi^-\rangle\langle\Phi^-| = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle + |1\rangle|0\rangle\right) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \Rightarrow \Psi^+ = |\Psi^+\rangle\langle\Psi^+| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle|1\rangle - |1\rangle|0\rangle\right) = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \Rightarrow \Psi^- = |\Psi^-\rangle\langle\Psi^-| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Add up: $\Phi^+ + \Phi^- + \Psi^+ + \Psi^- = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$

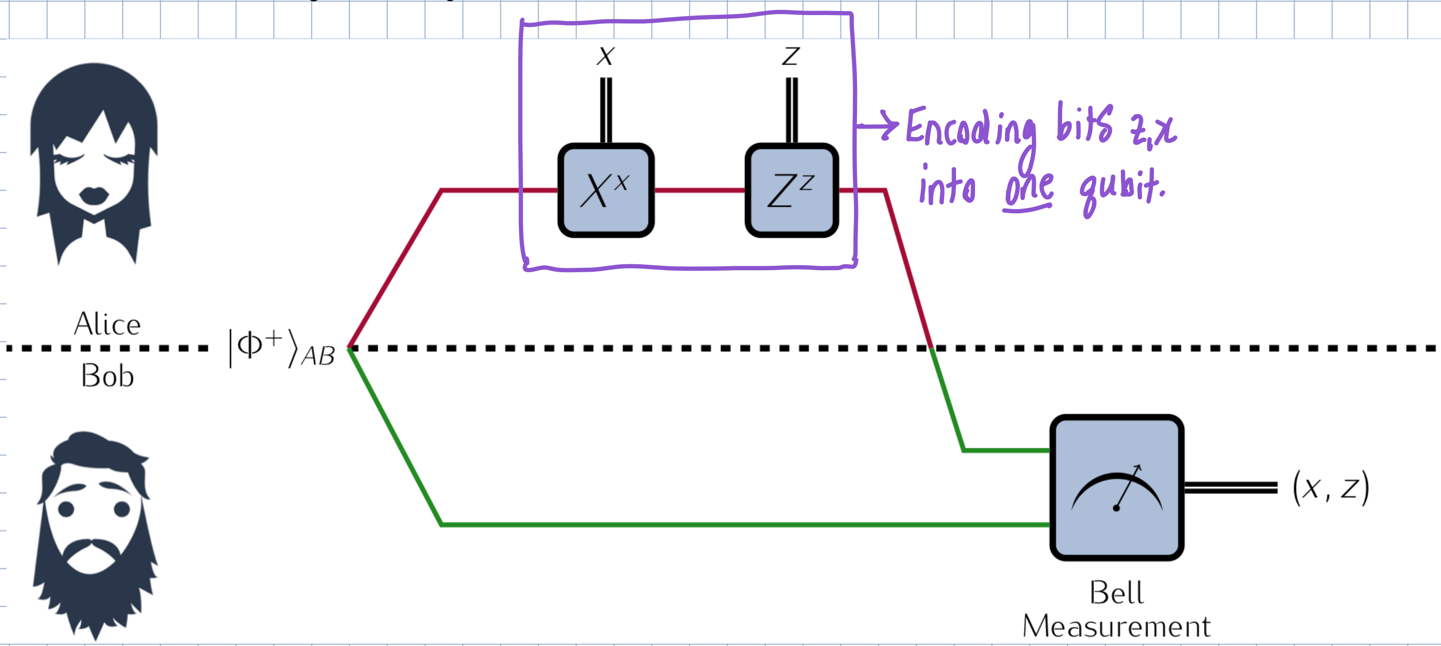
$$+ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \blacksquare$$

② Superdense Coding

⊛ Teleportation: using entanglement + classical information to transmit quantum information.

⊛ We can also use entanglement to transmit classical information!

- Superdense coding: using entanglement + 1 qubit to transmit 2 classical bits.



→ Encoding bits z,x into one qubit.

(i) • $\underbrace{\text{To encode } (0,0)}_{(z,x)}$: $(\underbrace{Z^0X^0}_{=\mathbb{1}} \otimes \mathbb{1})|\Phi^+\rangle_{AB} = |\Phi^+\rangle_{AB}$ (Alice does nothing).

• To encode $(0,1)$: $(\underbrace{Z^0X^1}_{=X} \otimes \mathbb{1})|\Phi^+\rangle_{AB} = (X \otimes \mathbb{1})\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$

$= \frac{1}{\sqrt{2}}(X|0\rangle_A|0\rangle_B + X|1\rangle_A|1\rangle_B)$

(Alice applies X to her qubit.)

$= \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)$

$= |\Psi^+\rangle_{AB}$

• To encode $(1,0)$: $(\underbrace{Z^1X^0}_{=Z} \otimes \mathbb{1})|\Phi^+\rangle_{AB} = (Z \otimes \mathbb{1})\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$

(Alice applies Z to her qubit.)

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\left(Z|0\rangle_A |0\rangle_B + Z|1\rangle_A |1\rangle_B \right)$$

$$= \frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B - |1\rangle_A |1\rangle_B \right)$$

$$= |\Phi^-\rangle_{AB}$$

- To encode (1,1): $(\underbrace{Z^1 X^1}_{=ZX} \otimes \mathbb{1})|\Phi^+\rangle_{AB} = (ZX \otimes \mathbb{1})\frac{1}{\sqrt{2}}\left(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B\right)$

$$= \frac{1}{\sqrt{2}}\left(\underbrace{ZX|0\rangle_A}_{= Z|1\rangle = -|1\rangle} |0\rangle_B + \underbrace{ZX|1\rangle_A}_{= Z|0\rangle = |0\rangle} |1\rangle_B \right)$$

$$= \frac{1}{\sqrt{2}}\left(-|1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B \right)$$

$$= |\Psi^-\rangle_{AB}.$$

⊛ Observation: These encoded states are the same as the measurement operators of the Bell measurement!

(2) After encoding, Alice sends her qubit to Bob.

(3) Then Bob does a Bell measurement on both qubits.

⊛ Recall: the outcome of the Bell measurement is two bits of information.

Outcome $\Phi^+ \leftrightarrow (0,0)$

Outcome $\Psi^+ \leftrightarrow (0,1)$

Outcome $\Phi^- \leftrightarrow (1,0)$

Outcome $\Psi^- \leftrightarrow (1,1)$

Each one of these outcomes occurs with some probability, depending on the state: if the state being measured is ρ, then:

$$\Pr[\Phi^+] = \mathrm{Tr}[\rho\Phi^+],\ \Pr[\Phi^-] = \mathrm{Tr}[\rho\Phi^-],\ \Pr[\Psi^+] = \mathrm{Tr}[\rho\Psi^+],\ \Pr[\Psi^-] = \mathrm{Tr}[\rho\Psi^-]$$

⊛ The state encoded by Alice is

$|\Phi_{z,x}\rangle = (Z^z X^x \otimes \mathbb{1})|\Phi^+\rangle$. ($(z,x) \in \{0,1\}^2$ are the bits she wants to send to Bob.)

$$\left.\begin{aligned} |\Phi_{0,0}\rangle &= |\Phi^+\rangle \\ |\Phi_{0,1}\rangle &= |\Psi^+\rangle \\ |\Phi_{1,0}\rangle &= |\Phi^-\rangle \\ |\Phi_{1,1}\rangle &= |\Psi^-\rangle \end{aligned}\right\}$$

⊛ Bob measures one of these states, but he does not know which one it is. He has to figure that out from the Bell measurement.

⊛ What is the probability that Bob retrieves the bits Alice sent?

We need to determine: Pr[Bob gets outcome (z',x'), given Alice encoded (z,x)].

This is given by $\mathrm{Tr}[\underbrace{\Phi^{z',x'}}_{\text{Bob's measurement}} \underbrace{\Phi^{z,x}}_{\text{Alice's encoded state.}}]$ $\quad (\Phi^{z,x} = |\Phi^{z,x}\rangle\langle\Phi^{z,x}|.)$

$$\delta_{x,x'} = \begin{cases} 0 & \text{if } x \neq x' \\ 1 & \text{if } x = x' \end{cases}$$

We will show that $\boxed{\mathrm{Tr}[\Phi^{z',x'}\Phi^{z,x}] = \delta_{z,z'}\,\delta_{x,x'}}$

This means: if Alice sends bits z,x, then Bob's measurement outcome is also z,x, with probability one $\Rightarrow$ he successfully gets Alice's bits.

Proof: $\mathrm{Tr}[\Phi^{z',x'}\Phi^{z,x}] = \mathrm{Tr}\Big[\underbrace{\underbrace{(Z^{z'}X^{x'}\otimes\mathbb{1})|\Phi^+\rangle}_{|v_1\rangle}\underbrace{\langle\Phi^+|(X^{x'}Z^{z'}\otimes\mathbb{1})}_{\langle v_1|}}_{\Phi^{z',x'}}\underbrace{\underbrace{(Z^{z}X^{x}\otimes\mathbb{1})|\Phi^+\rangle}_{|v_2\rangle}\underbrace{\langle\Phi^+|(X^{x}Z^{z}\otimes\mathbb{1})}_{\langle v_2|}}_{\Phi^{z,x}}\Big].$

$$= \mathrm{Tr}[|v_1\rangle\underbrace{\langle v_1|v_2\rangle}_{\text{scalar!}}\langle v_2|] = \langle v_1|v_2\rangle \underbrace{\mathrm{Tr}[|v_1\rangle\langle v_2|]}_{=\langle v_2|v_1\rangle} = \langle v_1|v_2\rangle\underbrace{\langle v_2|v_1\rangle}_{=\overline{\langle v_1|v_2\rangle}}.$$

$(M_1\otimes\mathbb{1})(M_2\otimes\mathbb{1}) = M_1M_2\otimes\mathbb{1}$

$$\langle v_1|v_2\rangle = \langle\Phi^+|(X^{x'}Z^{z'}\otimes\mathbb{1})(Z^zX^x\otimes\mathbb{1})|\Phi^+\rangle = \langle\Phi^+|(X^{x'}Z^{z'}Z^zX^x\otimes\mathbb{1})|\Phi^+\rangle$$

$$= \frac{1}{\sqrt{2}}\underbrace{\left(\langle 0|\langle 0| + \langle 1|\langle 1|\right)}_{\sum_{k=0}^{1}\langle k|\langle k|}\left(X^{x'}Z^{z'}Z^{z}X^{x}\otimes\mathbb{1}\right)\frac{1}{\sqrt{2}}\underbrace{\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right)}_{\sum_{k'=0}^{1}|k'\rangle|k'\rangle}$$

$$= \frac{1}{2}\sum_{k,k'=0}^{1}\langle k|\langle k|\left(X^{x'}Z^{z'}Z^{z}X^{x}\otimes\mathbb{1}\right)|k'\rangle|k'\rangle$$

$$\langle k|\langle k|\left(M_1\otimes M_2\right)|k'\rangle|k'\rangle = \langle k|M_1|k'\rangle\langle k|M_2|k'\rangle$$

$$= \frac{1}{2}\sum_{k,k'=0}^{1}\langle k|X^{x'}Z^{z'}Z^{z}X^{x}|k'\rangle\underbrace{\langle k|k'\rangle}_{=\delta_{k,k'}}$$

$$= \frac{1}{2}\sum_{k=0}^{1}\langle k|X^{x'}Z^{z'}Z^{z}X^{x}|k\rangle \qquad \sum_{k}\langle k|M|k\rangle = Tr[M]$$

$$= \frac{1}{2}Tr\left[X^{x'}Z^{z'}Z^{z}X^{x}\right] = \frac{1}{2}Tr\left[\underbrace{X^{x}X^{x'}}Z^{z'}Z^{z}\right] = \underline{\underline{\frac{1}{2}Tr\left[X^{x\oplus x'}Z^{z'\oplus z}\right]}}.$$

$X^0 = \mathbb{1}$

$X^1 = X$

$$\left.\begin{aligned} x=0, x'=0 &\Rightarrow X^0X^0 = \mathbb{1}\\ x=0, x'=1 &\Rightarrow X^0X^1 = X\\ x=1, x'=0 &\Rightarrow X^1X^0 = X\\ x=1, x'=1 &\Rightarrow X^1X^1 = X^2 = \mathbb{1}\end{aligned}\right\} \Rightarrow X^xX^{x'} = X^{\underline{\underline{x\oplus x'}}}$$

$\oplus$

XOR!

$$\begin{pmatrix} 0\oplus 0 = 0\\ 0\oplus 1 = 1\\ 1\oplus 0 = 1\\ 1\oplus 1 = 0\end{pmatrix}$$

Same for $Z^{z'}Z^{z}$:
$$\left.\begin{aligned} z'=0, z=0 &\Rightarrow \mathbb{1}\\ z'=0, z=1 &\Rightarrow Z\\ z'=1, z=0 &\Rightarrow Z\\ z'=1, z=1 &\Rightarrow \mathbb{1}\end{aligned}\right\} \Rightarrow Z^{z'}Z^{z} = Z^{z'\oplus z}$$

Now, we use the fact that Z and X are orthogonal:

$$Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \Rightarrow Tr[X] = Tr[Z] = 0, \text{ and}$$
$$Tr[ZX] = Tr\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = Tr\left[\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right] = 0.$$

Therefore, $\mathrm{Tr}[Z^a X^b] = 2\,\delta_{a,0}\,\delta_{b,0}$. $\quad a, b \in \{0,1\}$

Check: $a=0, b=0 \Rightarrow \mathrm{Tr}[\mathbb{1}] = 2$

$a=0, b=1 \Rightarrow \mathrm{Tr}[X] = 0$

$a=1, b=0 \Rightarrow \mathrm{Tr}[Z] = 0$

$a=1, b=1 \Rightarrow \mathrm{Tr}[ZX] = 0$.

Finally: $\langle v_1 | v_2 \rangle = \frac{1}{2}\mathrm{Tr}\big[X^{\overbrace{x\oplus x'}^{a}} Z^{\overbrace{z'\oplus z}^{b}}\big] = \delta_{x\oplus x',0}\,\delta_{z'\oplus z,0}$

But $x \oplus x' = 0 \Leftrightarrow x = x'$ and $z' \oplus z = 0 \Leftrightarrow z' = z$

$\Rightarrow \delta_{x\oplus x',0} = \delta_{x,x'}$ and $\delta_{z'\oplus z,0} = \delta_{z',z}$.

So $\langle v_1 | v_2 \rangle = \delta_{x,x'}\,\delta_{z,z'}$.

Similarly $\langle v_2 | v_1 \rangle = \overline{\langle v_1 | v_2 \rangle} = \delta_{x,x'}\,\delta_{z,z'}$.

So $\mathrm{Tr}[\Phi^{z',x'}\Phi^{z,x}] = \delta_{x,x'}\,\delta_{z,z'}$. ∎

⊛ Recap: Superdense coding allows Alice to send two bits to Bob using only one qubit — as long as the qubit is already entangled with Bob's qubit.

This worked b/c the 4 Bell states were used to encode the information and do the measurement — and b/c the Bell states are orthogonal, they are perfectly distinguishable.

⊛ Why is this interesting?

With just a single (unentangled) qubit, we cannot successfully (w/ prob. 1) transmit more than one bit of information.

In dimension d, we can transmit w/ prob. 1 at most $\log_2(d)$ bits.

So $d=2 \Rightarrow \log_2(2)=1$ bit.

With entanglement, the dimension effectively increases to $4 \rightarrow \log_2(4)=2$.

(b/c $4=2^2$).