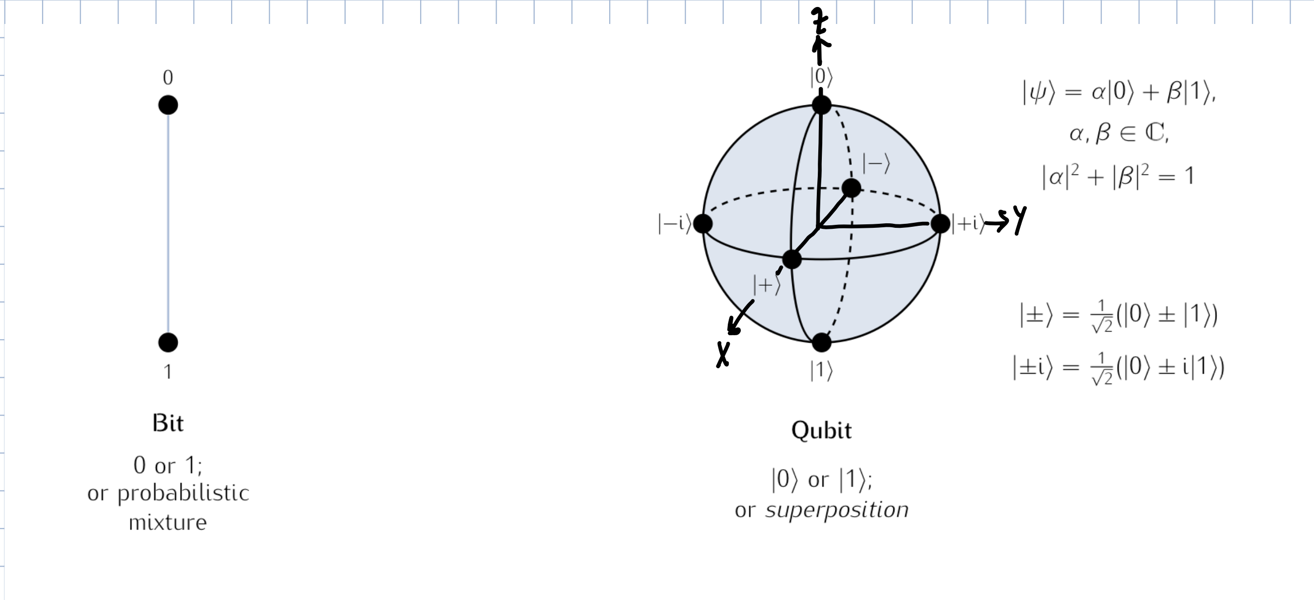
① Recap of the course so far.

(a) What is a qubit?



- The state of a qubit is given by a density matrix/operator:

  (1) $\rho^\dagger = \rho$ (Hermitian). Conjugate transpose

  (2) $\rho \geq 0$ (positive semi-definite $\Leftrightarrow$ non-negative eigenvalues).

  (3) $\mathrm{Tr}[\rho] = 1$ Sum of the diagonal elements.

- Points on the surface of the Bloch sphere are given by state vectors; they are called pure states: $\rho = |\psi\rangle\langle\psi|$, $|\psi\rangle \in \mathbb{C}^2$, $\| |\psi\rangle \|_2 = 1$.

  State vector.

- Points inside the Bloch sphere are mixed states:

$$\rho = \sum_{k=1}^{r} q_k \underbrace{|\psi_k\rangle\langle\psi_k|}_{\text{pure states}}, \quad q_k \in [0,1], \quad \sum_{k=1}^{r} q_k = 1.$$

- Quantum gates are used to describe operations/computations on qubits. They are defined by unitary matrices: $U \in L(\mathbb{C}^d)$, $U^\dagger U = UU^\dagger = \mathbb{1}$.

(b) How to extract information from a qubit? Measurements!

- A measurement (aka "positive operator-valued measure" (POVM)) is defined by a set of operators $\{M_x\}_{x\in\mathcal{X}}$ (some finite set $\mathcal{X}$), such that:

  (1) $M_x \geq 0 \ \forall x \in \mathcal{X}$. ($M_x \in L(\mathbb{C}^d) \to d\times d$ matrix for arbitrary dimension d.)

  (2) $\sum_{x\in\mathcal{X}} M_x = \mathbb{1}$

- For any state ρ, the outcome probabilities are given by
  $\Pr[x] = \mathrm{Tr}[M_x \rho] \ \forall x \in \mathcal{X}$.

- Examples of measurements:

  (1) Pauli-Z / computational basis: $\{\underbrace{|0\rangle\langle 0|}_{M_0}, \underbrace{|1\rangle\langle 1|}_{M_1}\}$ $\quad |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$. $\quad Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$

  (2) Pauli-X: $\{|+\rangle\langle +|, |-\rangle\langle -|\}$, $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, $X|\pm\rangle = \pm|\pm\rangle$

  (3) Any basis: for any unitary U, $\{U|0\rangle\langle 0|U^\dagger, U|1\rangle\langle 1|U^\dagger\}$.

  (Check: $U|0\rangle\langle 0|U^\dagger + U|1\rangle\langle 1|U^\dagger = U\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{=\mathbb{1}}U^\dagger = UU^\dagger = \mathbb{1}$ ✓)

(c) Entanglement: One of the key distinguishing characteristics of quantum vs. classical.

- A state vector $|\psi\rangle_{AB}$ is entangled if it cannot be written as $\underbrace{|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\varphi\rangle_B}_{\text{product state}}$.

- Every state vector $|\psi\rangle_{AB}$ has a Schmidt decomposition:

  (1) $|\psi\rangle_{AB} = \sum_{k=1}^{r} s_k |e_k\rangle_A \otimes |f_k\rangle_B$ $\qquad |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$.

  (2) s_k: Schmidt coefficients $(s_k > 0)$

  $\{|e_k\rangle_A\}_{k=1}^{r}$ and $\{|f_k\rangle\}_{k=1}^{r}$ are orthonormal vectors.

  $\langle e_k|e_{k'}\rangle = \delta_{k,k'} = \begin{cases} 0 & \text{if } k \neq k' \\ 1 & \text{if } k = k' \end{cases}$

r : Schmidt rank.

(3) $|\psi\rangle_{AB}$ is entangled if and only if $r > 1$.

- Entanglement of mixed states is more complicated!

A mixed state (density operator) ρ_{AB} is entangled if cannot be written as

$$\rho_{AB} = \sum_{x \in X} \underbrace{p(x)}_{\text{probabilities}} \tau_A^{(x)} \otimes \omega_B^{(x)} \rightarrow \text{This is separable.}$$

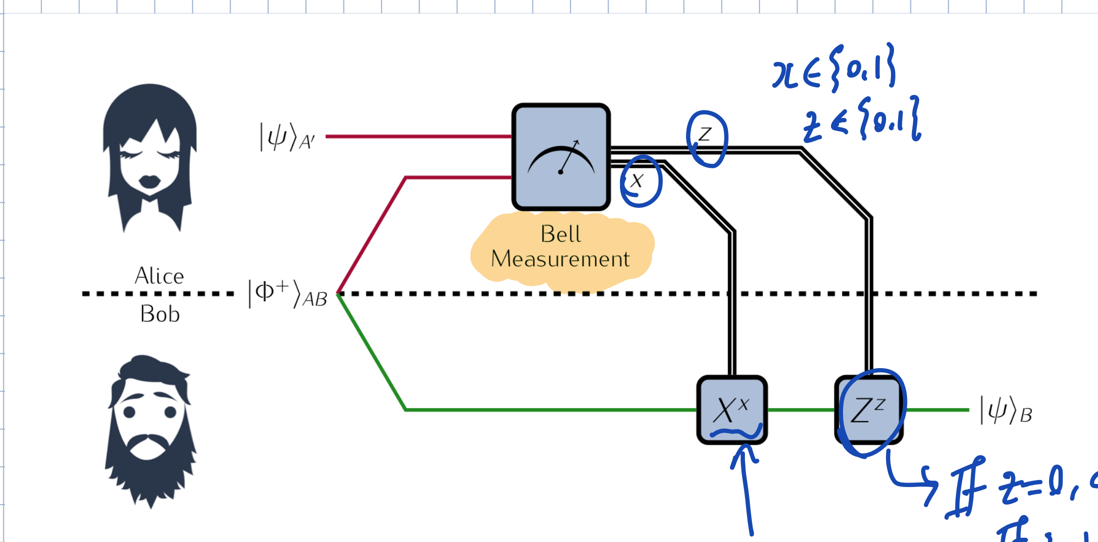
(*) If A + B are qubits, then ρ_{AB} is separable if and only if $\rho_{AB}^{T_B} \geq 0$. (partial transpose)

② Teleportation

- A method to transfer the state of a qubit from Alice to Bob using entanglement and classical communication only.

- Given: (1) State vector that Alice wants to send to Bob.
  (2) Shared entangled state b/w Alice and Bob.

- Goal : Transfer the state of qubit A′ to Bob's qubit.



If $z=0$, do nothing
If $z=1$, apply Z.

If $x=0$ do nothing
If $x=1$, apply X.

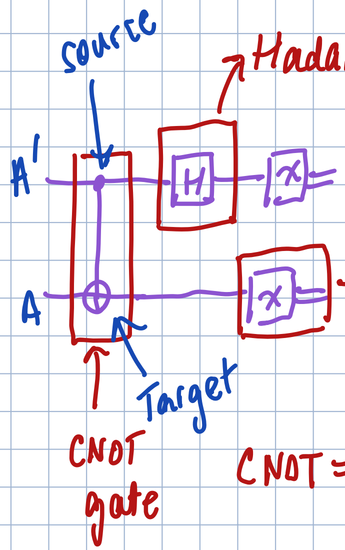
(a) Key ingredient: the two-qubit Bell measurement

- POVM is $\{\Phi^+, \Phi^-, \Psi^+, \Psi^-\}$, $\Phi^\pm = |\Phi^\pm\rangle\langle\Phi^\pm|$, $\Psi^\pm = |\Psi^\pm\rangle\langle\Psi^\pm|$.

$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle \pm |1\rangle|1\rangle)$

$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle \pm |1\rangle|0\rangle)$.

- Circuit description of the measurement:



$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

source target

$CNOT|0,0\rangle = |0,0\rangle$

$CNOT|0,1\rangle = |0,1\rangle$

$CNOT|1,0\rangle = |1,1\rangle$

$CNOT|1,1\rangle = |1,0\rangle$

(b) Protocol analysis

$|\alpha|^2 + |\beta|^2 = 1$

$|\psi\rangle_{A'} = \alpha|0\rangle_{A'} + \beta|1\rangle_{A'}$ → State Alice wants to send to Bob.

$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0,0\rangle_{AB} + |1,1\rangle_{AB})$ → Shared entanglement.

(1) Joint initial state: $|\psi\rangle_{A'} \otimes |\Phi^+\rangle_{AB} = (\alpha|0\rangle_{A'} + \beta|1\rangle_{A'}) \otimes \frac{1}{\sqrt{2}}(|0,0\rangle_{AB} + |1,1\rangle_{AB})$

$$= \frac{1}{\sqrt{2}}\big(\alpha|0,0,0\rangle_{A'AB} + \alpha|0,1,1\rangle_{A'AB} + \beta|1,0,0\rangle_{A'AB} + \beta|1,1,1\rangle_{A'AB}\big)$$

(2) $CNOT_{A'A}$: $\frac{1}{\sqrt{2}}\Big(\alpha|0,0,0\rangle_{A'AB} + \alpha|0,1,1\rangle_{A'AB}$

$+\beta|\underbrace{1,0}_{1,1},0\rangle_{A'AB} + \beta|\underbrace{1,1}_{1,0},1\rangle_{A'AB}\Big)$

$\downarrow$

$$\frac{1}{\sqrt{2}}\Big(\alpha|0,0,0\rangle_{A'AB} + \alpha|0,1,1\rangle_{A'AB} + \beta|1,1,0\rangle_{A'AB} + \beta|1,0,1\rangle_{A'AB}\Big)$$

(3) Hadamard on A' : $\frac{1}{\sqrt{2}}\Big(\alpha|\overset{|+\rangle}{0},0,0\rangle_{A'AB} + \alpha|\overset{|+\rangle}{0},1,1\rangle_{A'AB}$

$+\beta|\underset{|-\rangle}{1},1,0\rangle_{A'AB} + \beta|\underset{|-\rangle}{1},0,1\rangle_{A'AB}\Big)$

$H|0\rangle = |+\rangle$

$H|1\rangle = |-\rangle$

$\downarrow$

$$\frac{1}{\sqrt{2}}\Big(\alpha|+,0,0\rangle_{A'AB} + \alpha|+,1,1\rangle_{A'AB} + \beta|-,1,0\rangle_{A'AB} + \beta|-,0,1\rangle_{A'AB}\Big)$$

(4) Measure $A'A$ in the Z-basis (computational basis) $\{|0\rangle,|1\rangle\}$ basis:

- <u>Four outcomes</u>: 00, 01, 10, 11 $= (z,x)$

$$\frac{1}{\sqrt{2}}\Big(\alpha|+,0,0\rangle_{A'AB} + \alpha|+,1,1\rangle_{A'AB} + \beta|-,1,0\rangle_{A'AB} + \beta|-,0,1\rangle_{A'AB}\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\alpha\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)_{A'}|0,0\rangle_{AB} + \alpha\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)_{A'}|1,1\rangle_{AB}$$

$$+ \beta\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)_{A'}|1,0\rangle_{AB} + \beta\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)_{A'}|0,1\rangle_{AB}\Big)$$

$\downarrow$

$$|\phi\rangle_{A'AB} = \Big(|0,0\rangle_{A'A}\frac{1}{2}(\alpha|0\rangle+\beta|1\rangle)_B + |0,1\rangle_{A'A}\frac{1}{2}(\alpha|1\rangle+\beta|0\rangle)_B$$

$$+|1,0\rangle_{A'A}\frac{1}{2}(\alpha|0\rangle-\beta|1\rangle)_B + |1,1\rangle_{A'A}\frac{1}{2}(\alpha|1\rangle-\beta|0\rangle)_B\Big)$$

$\{|0,0\rangle\langle 0,0|,\ |0,1\rangle\langle 0,1|,\ |1,0\rangle\langle 1,0|,\ |1,1\rangle\langle 1,1|\}$

• <u>Outcome $(0,0)$</u>: $\Pr[0,0] = \mathrm{Tr}\left[\left(|0,0\rangle\langle 0,0|_{A'A} \otimes \mathbb{1}_B\right)|\phi\rangle\langle\phi|_{A'AB}\right]$

$$= \mathrm{Tr}\left[\left(\langle 0,0|_{A'A} \otimes \mathbb{1}_B\right)|\phi\rangle\langle\phi|_{A'AB}\left(|0,0\rangle_{A'A} \otimes \mathbb{1}_B\right)\right]$$

$$\left(\langle 0,0|_{A'A} \otimes \mathbb{1}_B\right)|\phi\rangle_{A'AB} = \left(\langle 0,0|_{A'A} \otimes \mathbb{1}_B\right)\frac{1}{2}\Big(|0,0\rangle_{A'A}\left(\alpha|0\rangle+\beta|1\rangle\right)_B + |0,1\rangle_{A'A}\left(\alpha|1\rangle+\beta|0\rangle\right)_B$$
$$+ |1,0\rangle_{A'A}\left(\alpha|0\rangle-\beta|1\rangle\right)_B + |1,1\rangle_{A'A}\left(\alpha|1\rangle-\beta|0\rangle\right)_B\Big)$$

$$= \frac{1}{2}\Big(\underbrace{\langle 0,0|0,0\rangle_{A'A}}_{1}\left(\alpha|0\rangle+\beta|1\rangle\right)_B + \underbrace{\langle 0,0|0,1\rangle_{A'A}}_{0}\left(\alpha|1\rangle+\beta|0\rangle\right)_B$$
$$+ \underbrace{\langle 0,0|1,0\rangle_{A'A}}_{0}\left(\alpha|0\rangle-\beta|1\rangle\right)_B + \underbrace{\langle 0,0|1,1\rangle_{A'A}}_{0}\left(\alpha|1\rangle-\beta|0\rangle\right)_B\Big)$$

$$= \frac{1}{2}\left(\alpha|0\rangle_B+\beta|1\rangle\right)$$

$$= \frac{1}{2}|\psi\rangle_B \Rightarrow \Pr[0,0] = \mathrm{Tr}\left[\frac{1}{2}|\psi\rangle\langle\psi|\frac{1}{2}\right] = \frac{1}{4}\underbrace{\mathrm{Tr}\left[|\psi\rangle\langle\psi|\right]}_{=1} = \frac{1}{4}.$$

State of Bob conditioned on outcome $(0,0)$ is $|\psi\rangle$!
(Exactly the state Alice wanted to send.)

• <u>Outcome $(0,1)$</u>: $\Pr[0,1] = \mathrm{Tr}\left[\left(|0,1\rangle\langle 0,1|_{A'A} \otimes \mathbb{1}_B\right)|\phi\rangle\langle\phi|_{A'AB}\right]$

$$= \mathrm{Tr}\left[\left(\langle 0,1| \otimes \mathbb{1}_B\right)|\phi\rangle\langle\phi|_{A'AB}\left(|0,1\rangle_{A'A} \otimes \mathbb{1}_B\right)\right].$$

$$|\phi\rangle_{A'AB} = \frac{1}{2}\Big(|0,0\rangle_{A'A}\left(\alpha|0\rangle+\beta|1\rangle\right)_B + |0,1\rangle_{A'A}\left(\alpha|1\rangle+\beta|0\rangle\right)_B$$
$$+ |1,0\rangle_{A'A}\left(\alpha|0\rangle-\beta|1\rangle\right)_B + |1,1\rangle_{A'A}\left(\alpha|1\rangle-\beta|0\rangle\right)_B\Big)$$

$|\psi\rangle = \alpha|0\rangle+\beta|1\rangle$

$X|\psi\rangle = \alpha X|0\rangle + \beta X|1\rangle = \alpha|1\rangle+\beta|0\rangle$

$$\left(\langle 0,1|_{A'A} \otimes \mathbb{1}_B\right)|\phi\rangle_{A'AB} = \frac{1}{2}\left(\alpha|1\rangle+\beta|0\rangle\right) = \frac{1}{2}X|\psi\rangle_B.$$ Pauli X.

$$\mathrm{Tr}\left[\frac{1}{2}X|\psi\rangle\langle\psi|X\frac{1}{2}\right] = \frac{1}{4}\mathrm{Tr}\left[X|\psi\rangle\langle\psi|X\right] = \frac{1}{4}\mathrm{Tr}\left[|\psi\rangle\langle\psi|\right] = \frac{1}{4}.$$

$\Rightarrow \Pr[0,1] = \frac{1}{4}$ $\qquad X^2 = \mathbb{1}$

$\Rightarrow$ State of Bob conditioned on Alice's outcome is $X|\psi\rangle$.

- <u>Outcome $(1,0)$</u>: $\Pr(1,0) = \mathrm{Tr}\left[\left(\langle 1,0|_{A'A} \otimes \mathbb{1}_B\right)|\phi\rangle\langle\phi|_{A'AB}\left(|1,0\rangle_{A'A} \otimes \mathbb{1}_B\right)\right]$

$$|\phi\rangle_{A'AB} = \frac{1}{2}\Big(|0,0\rangle_{A'A}\left(\alpha|0\rangle + \beta|1\rangle\right)_B + |0,1\rangle_{A'A}\left(\alpha|1\rangle + \beta|0\rangle\right)_B$$
$$+ |1,0\rangle_{A'A}\left(\alpha|0\rangle - \beta|1\rangle\right)_B + |1,1\rangle_{A'A}\left(\alpha|1\rangle - \beta|0\rangle\right)_B\Big)$$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$Z|\psi\rangle = \alpha Z|0\rangle + \beta Z|1\rangle$ $\quad Z|1\rangle = -|1\rangle$

$= \alpha|0\rangle - \beta|1\rangle$.

$$\left(\langle 1,0|_{A'A} \otimes \mathbb{1}_B\right)|\phi\rangle_{A'AB} = \frac{1}{2}\left(\alpha|0\rangle - \beta|1\rangle\right) = \frac{1}{2}Z|\psi\rangle$$

Pauli Z

$$\mathrm{Tr}\left[\frac{1}{2}Z|\psi\rangle\langle\psi|Z\frac{1}{2}\right] = \frac{1}{4}\mathrm{Tr}\left[Z|\psi\rangle\langle\psi|Z\right] = \frac{1}{4}\mathrm{Tr}\left[|\psi\rangle\langle\psi|\right] = \frac{1}{4}.$$

$Z^2 = \mathbb{1}$

$\Rightarrow \Pr(1,0) = \frac{1}{4}$

$\Rightarrow$ State of Bob conditioned on Alice's outcome is $Z|\psi\rangle$.

- <u>Outcome $(1,1)$</u>: $\Pr(1,1) = \mathrm{Tr}\left[\left(\langle 1,1|_{A'A} \otimes \mathbb{1}_B\right)|\phi\rangle\langle\phi|_{A'AB}\left(|1,1\rangle_{A'A} \otimes \mathbb{1}_B\right)\right]$

$$|\phi\rangle_{A'AB} = \frac{1}{2}\Big(|0,0\rangle_{A'A}\left(\alpha|0\rangle + \beta|1\rangle\right)_B + |0,1\rangle_{A'A}\left(\alpha|1\rangle + \beta|0\rangle\right)_B$$
$$+ |1,0\rangle_{A'A}\left(\alpha|0\rangle - \beta|1\rangle\right)_B + |1,1\rangle_{A'A}\left(\alpha|1\rangle - \beta|0\rangle\right)_B\Big)$$

$$\left(\langle 1,1|_{A'A} \otimes \mathbb{1}_B\right)|\phi\rangle_{A'AB} = \frac{1}{2}\left(\alpha|1\rangle - \beta|0\rangle\right) = \frac{1}{2}XZ|\psi\rangle_B$$

$$\mathrm{Tr}\left[\frac{1}{2}XZ|\psi\rangle\langle\psi|ZX\frac{1}{2}\right] = \frac{1}{4}\mathrm{Tr}\left[ZX|\psi\rangle\langle\psi|XZ\right] = \frac{1}{4}\mathrm{Tr}\left[|\psi\rangle\langle\psi|\right] = \frac{1}{4}.$$

$Z^2 = \mathbb{1},\ X^2 = \mathbb{1}$

$\Rightarrow \Pr(1,1) = \frac{1}{4}$

$\Rightarrow$ State of Bob conditioned on Alice's outcome is $XZ|\psi\rangle$.

<u>Summary:</u>

- $Pr(0,0) = \frac{1}{4} \rightarrow$ State $|\psi\rangle$ for Bob.
- $Pr(0,1) = \frac{1}{4} \rightarrow$ State $X|\psi\rangle$ for Bob.
- $Pr(1,0) = \frac{1}{4} \rightarrow$ State $Z|\psi\rangle$ for Bob.
- $Pr(1,1) = \frac{1}{4} \rightarrow$ State $XZ|\psi\rangle$ for Bob.

(5) After the measurement, Alice communicates the outcomes to Bob.

(6) Depending on the outcome, Bob applies a <u>correction</u>:

z x

0,0 $\rightarrow$ No correction

0,1 $\rightarrow$ Apply Pauli-X $\quad X(X|\psi\rangle) = \underbrace{X^2}_{\mathbb{1}}|\psi\rangle = |\psi\rangle$

1,0 $\rightarrow$ Apply Pauli-Z $\quad Z(Z|\psi\rangle) = \underbrace{Z^2}_{=\mathbb{1}}|\psi\rangle = |\psi\rangle$

1,1 $\rightarrow$ Apply Pauli-X, then Pauli-Z

Then Bob recovers Alice's state!

⊛ Teleportation is <u>not</u> instantaneous / faster than light — it only works if Bob gets Alice's measurement outcomes $\rightarrow$ this takes time!

⊛ The teleported state can also be a mixed state.