① Recap: Entanglement

⊛ State vectors of two qubits belongs to the tensor-product space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

(a) – A **product state (vector)** has the form:

(State vector) $|\psi\rangle_{AB} = |\phi_1\rangle_A \otimes |\phi_2\rangle_B$, $|\phi_1\rangle \in \mathbb{C}^{d_1}$, $|\phi_2\rangle \in \mathbb{C}^{d_2}$.

(Density operator). $\rho_{AB} = \sigma_A \otimes \tau_B$, $\sigma_A \in L(\mathbb{C}^{d_A})$ and $\tau_B \in L(\mathbb{C}^{d_B})$ are density operators.

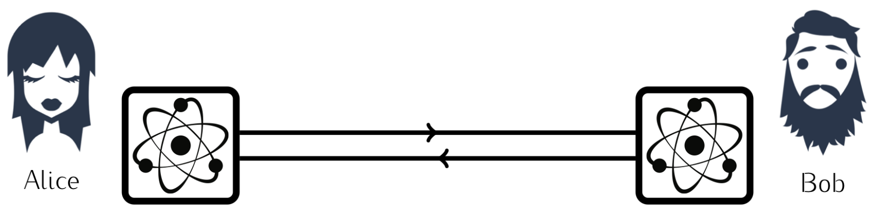
$$\rho_{AB} = \sigma_A \otimes \tau_B$$

Product state

Alice and Bob individually prepare their systems.

– A **separable state** has the form: $\rho_{AB} = \sum_x p(x)\, \sigma_A^x \otimes \tau_B^x$

↳ Also called "classically correlated" ↳ probabilities.

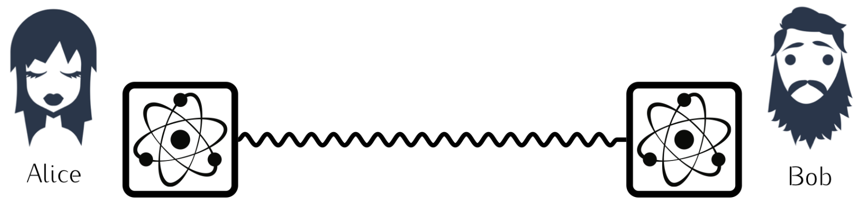


$$\rho_{AB} = \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \otimes \tau_B^x$$

Separable state

Alice and Bob individually prepare their systems via local operations and classical communication.

- An **entangled state** is NOT a separable state.



$$\rho_{AB} \neq \sum_{x\in\mathcal{X}} p(x)\sigma_A^x \otimes \tau_B^x$$

Entangled state

Correlations between Alice and Bob are non-local.
State of the individual systems not sufficient to describe the pair.

⊛ The **Bell states**

Pure states

$$\begin{cases} \Phi^\pm = |\Phi^\pm\rangle\langle\Phi^\pm|, & |\Phi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle \pm |1,1\rangle\right) \\ \Psi^\pm = |\Psi^\pm\rangle\langle\Psi^\pm|, & |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle \pm |1,0\rangle\right). \end{cases}$$

$$|\Phi^+\rangle = \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad |\Phi^-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad |\Psi^+\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad |\Psi^-\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

Observe that $\{|\Phi^\pm\rangle, |\Psi^\pm\rangle\} \longleftrightarrow \{ \underbrace{(Z^z X^x \otimes \mathbb{1})|\Phi^+\rangle}_{|\Phi_{z,x}\rangle} : z,x \in \{0,1\}\}$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z^0 = \mathbb{1},\ Z^1 = Z, \quad X^0 = \mathbb{1},\ X^1 = X$$

$0,0 \leftrightarrow \Phi^+$
$0,1 \leftrightarrow \Psi^+$
$1,0 \leftrightarrow \Phi^-$
$1,1 \leftrightarrow \Psi^-$

$|\Phi^-\rangle = (Z \otimes \mathbb{1})|\Phi^+\rangle$
$|\Psi^+\rangle = (X \otimes \mathbb{1})|\Phi^+\rangle$
$|\Psi^-\rangle = (ZX \otimes \mathbb{1})|\Phi^+\rangle$

② Determining Entanglement.

⊛ Given a state vector $|\psi\rangle$, how to determine if it is entangled or not?

Precisely: $\overset{?}{\exists}\ |\phi_1\rangle, |\phi_2\rangle$ s.t. $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$?

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

(a) $|\psi\rangle_{AB} \in \mathbb{C}^d \otimes \mathbb{C}^d \rightarrow |\psi\rangle_{AB} = \sum_{i,j=0}^{d-1} M_{i,j}\, |i\rangle_A \otimes |j\rangle_B$

two-index quantity $\longleftrightarrow$ matrix!

Let $M = \sum_{i,j=0}^{d-1} M_{i,j}\, |j\rangle\langle i| \in L(\mathbb{C}^d)$.

$M|i\rangle = \sum_{j=0}^{d-1} M_{i,j}|j\rangle$

$d=2$: $|\Gamma_2\rangle = |0\rangle|0\rangle + |1\rangle|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Lemma: $|\psi\rangle_{AB} = (\mathbb{1} \otimes M)|\Gamma_d\rangle$, $\quad |\Gamma_d\rangle = \sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle$.

Proof: $(\mathbb{1} \otimes M)|\Gamma_d\rangle = (\mathbb{1} \otimes M)\left(\sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle\right) = \sum_{i=0}^{d-1} |i\rangle \otimes M|i\rangle = \sum_{i,j=0}^{d-1} M_{i,j}\, |i\rangle \otimes |j\rangle = |\psi\rangle_{AB}$. ■

$\mathrm{vec}(M) := (\mathbb{1} \otimes M)|\Gamma_d\rangle \equiv |M\rangle\rangle \Rightarrow$ Stacking the columns into a vector!

Example: $d=2$, $M = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & a & b \\ 1 & c & d \end{array}$

$\Rightarrow (\mathbb{1} \otimes M)|\Gamma_2\rangle = a|0,0\rangle + c|0,1\rangle + b|1,0\rangle + d|1,1\rangle$

$$= \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}$$

⊛ For every $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, $\exists\ M \in L(\mathbb{C}^d)$ such that $|\psi\rangle = \mathrm{vec}(M) = (\mathbb{1} \otimes M)|\Gamma\rangle$.

(b) Every $M \in L(\mathbb{C}^d)$ has a singular-value decomposition:

$$M = \underbrace{U}_{\text{unitaries}} \overbrace{\Sigma}^{\text{matrix of singular values}} \underbrace{V^\dagger}_{\text{unitaries}}$$

In bra-ket notation: $M = \sum_{k=1}^{r} s_k |f_k\rangle\langle e_k|$

singular values, $s_k > 0 \ \forall k$. (unique up to ordering)

rank $\equiv$ # of singular values $\rightarrow$ unique

$\{|f_k\rangle\}_k, \{|e_k\rangle\}_k$ are orthonormal (left and right singular vectors).

$$\Rightarrow |\psi\rangle_{AB} = \mathrm{vec}(M) = (\mathbb{1} \otimes M)|\Gamma_d\rangle = \sum_{k=1}^{r} s_k \, \mathrm{vec}(|f_k\rangle\langle e_k|) = \sum_{k=1}^{r} s_k \, |\overline{e_k}\rangle_A \otimes |f_k\rangle_B$$

$$= (\mathbb{1} \otimes |f_k\rangle\langle e_k|)\left(\sum_{i=0}^{d-1} |i\rangle \otimes |i\rangle\right) = \sum_{i=0}^{d-1} |i\rangle \otimes |f_k\rangle \underbrace{\langle e_k|i\rangle}$$

$$|e_k\rangle = \sum_{i=0}^{d-1} c_i |i\rangle \qquad c_i = \langle i|e_k\rangle$$

$$= \overline{\langle i|e_k\rangle} = \langle i|\overline{e_k}\rangle \qquad |\overline{e_k}\rangle = \sum_{i=0}^{d-1} \overline{c_i}|i\rangle$$

$$= \underbrace{\sum_{i=0}^{d-1} |i\rangle\langle i|}_{\mathbb{1}} \overline{e_k}\rangle \otimes |f_k\rangle = |\overline{e_k}\rangle \otimes |f_k\rangle$$

⊛ $|\psi\rangle_{AB} = \sum_{k=1}^{r} s_k |\overline{e_k}\rangle_A \otimes |f_k\rangle$ is called the Schmidt decomposition of $|\psi\rangle_{AB}$.

⊛ $\{s_k\}_{k=1}^{r}$ are called Schmidt coefficients. (Sometimes $\sqrt{s_k}$).

$\hookrightarrow s > 0$

(Normalisation: $\langle\psi|\psi\rangle = 1 \Rightarrow \sum_{k=1}^{r} s_k^2 = 1$)

$$1 = \langle\psi|\psi\rangle = \left(\sum_{k=1}^{r} s_k \langle\overline{e_k}|_A \otimes \langle f_k|\right)\left(\sum_{k'=1}^{r} s_{k'} |\overline{e_{k'}}\rangle \otimes |f_{k'}\rangle\right) = \sum_{k,k'=1}^{r} s_k s_{k'} \overbrace{\langle\overline{e_k}|\overline{e_{k'}}\rangle}^{\delta_{kk'}} \overbrace{\langle f_k|f_{k'}\rangle}^{\delta_{kk'}} = \sum_{k=1}^{r} s_k^2$$

- If $r=1$, then $|\psi\rangle = |e_1\rangle \otimes |f_1\rangle \Rightarrow$ not entangled!
- If $r>1$, then entangled! $r \equiv$ **Schmidt rank**.

(c) Observe: $|\psi\rangle_{AB} = \sum_{k=1}^{r} s_k |\overline{e_k}\rangle_A \otimes |f_k\rangle_B$

$$\Rightarrow Tr_B\left[|\psi\rangle\langle\psi|_{AB}\right] = Tr_B\left[\sum_{k,k'=1}^{r} s_k s_{k'} |\overline{e_k}\rangle\langle\overline{e_{k'}}|_A \otimes |f_k\rangle\langle f_{k'}|_B\right]$$

$$= \sum_{k,k'=1}^{r} s_k s_{k'} |\overline{e_k}\rangle\langle\overline{e_{k'}}|_A \underbrace{Tr\left[|f_k\rangle\langle f_{k'}|\right]}_{=\delta_{k,k'}}$$

$$= \sum_{k=1}^{r} s_k^2 |\overline{e_k}\rangle\langle\overline{e_k}|_A \rightarrow \text{Diagonal!}$$

Also, $Tr_A\left[|\psi\rangle\langle\psi|_{AB}\right] = \sum_{k=1}^{r} s_k^2 |f_k\rangle\langle f_k|_B$

$\Rightarrow Tr_A[\psi_{AB}]$ and $Tr_B[\psi_{AB}]$ have the same (non-zero) eigenvalues!

(d) Suppose $r=d$, $s_k = \frac{1}{\sqrt{d}}$ $\forall k \in \{1,2,\ldots,d\}$.

We call $|\psi\rangle$ **maximally entangled** (Note that the marginals are maximally mixed.).

(e) Observe: $\frac{1}{\sqrt{d}} \sum_{k=1}^{d} |e_k\rangle \otimes |\overline{e_k}\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \left(\sum_{j=0}^{d-1} |j\rangle\langle j|\right) |e_k\rangle \otimes \left(\sum_{\ell=0}^{d-1} |\ell\rangle\langle \ell|\right) |\overline{e_k}\rangle$

$$= |\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k,k\rangle$$

$$\frac{1}{\sqrt{2}}\left(|0,0\rangle + |1,1\rangle\right)$$

$$\frac{1}{\sqrt{2}}\left(\underbrace{|0\rangle}_{|\bar{e}_1\rangle}\underbrace{|1\rangle}_{|f_1\rangle} + \underbrace{|1\rangle}_{|\bar{e}_2\rangle}\underbrace{|0\rangle}_{|f_2\rangle}\right)$$

$$= \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \sum_{j,l=0}^{d-1} |j\rangle \otimes |l\rangle \langle j|e_k\rangle\langle l|\bar{e}_k\rangle \qquad \langle \bar{e}_k | l\rangle = \langle e_k | l\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \sum_{j,l=0}^{d-1} |j\rangle \otimes |l\rangle \langle j|e_k\rangle\langle e_k|l\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{j,l=0}^{d-1} |j\rangle \otimes |l\rangle \langle j| \underbrace{\left(\sum_{k=1}^{d} |e_k\rangle\langle e_k|\right)}_{= \mathbb{1}_d \text{ b/c } \{|e_k\rangle\} \text{ is ONB!}} |l\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{j,l=0}^{d-1} |j,l\rangle \langle j|l\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j,j\rangle$$

$$= \underline{\underline{|\Phi_d^+\rangle}}.$$

(f) Vectorized unitaries are maximally entangled

$$|\psi\rangle_{AB} = (\mathbb{1} \otimes \underset{\text{unitary}}{U}) |\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \underbrace{|k\rangle}_{\equiv |\bar{e}_k\rangle} \otimes \underbrace{U|k\rangle}_{\equiv |f_k\rangle}$$

Also,

$$\mathrm{Tr}_A\left[|\psi\rangle\langle\psi|_{AB}\right] = \mathrm{Tr}_A\left[(\mathbb{1}_A \otimes U_B)\underbrace{\Phi^+_{AB}}_{|\Phi_d^+\rangle\langle\Phi_d^+|_{AB}}(\mathbb{1}_A \otimes U_B^\dagger)\right]$$

$$= U \underbrace{\mathrm{Tr}_A\left[\Phi^+_{AB}\right]}_{= \frac{\mathbb{1}_B}{d}} U^\dagger \quad (*).$$

$$= \frac{1}{d} U U^\dagger$$

$$= \frac{1}{d}\mathbb{1}_B \qquad \left(\text{Similar for } \mathrm{Tr}_A\left[|\psi\rangle\langle\psi|_{AB}\right]\right).$$

$$= \sum_{k=0}^{d-1} (\langle k|_A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes U_B)\Phi^+(\mathbb{1}_A \otimes U_B^\dagger)(\mathbb{1} \otimes |k\rangle)$$

$$(M \otimes \mathbb{1})(\mathbb{1} \otimes N) = (\mathbb{1} \otimes N)(M \otimes \mathbb{1})$$

$$= \sum_{k=0}^{d-1} U_B \langle k|_A \Phi^+_{AB} |k\rangle U_B^\dagger = U_B \mathrm{Tr}_A\left[\Phi^+_{AB}\right] U_B^\dagger$$

Proof of (*): $\mathrm{Tr}_A[\Phi^+_{AB}] = \frac{1}{d}\mathrm{Tr}_A\left[\left(\sum_{k=0}^{d-1}|k\rangle_A|k\rangle_B\right)\left(\sum_{k'=0}^{d-1}\langle k'|_A\langle k'|_B\right)\right].$

$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k\rangle|k\rangle$

$$= \frac{1}{d}\sum_{k,k'=0}^{d-1}\underbrace{\mathrm{Tr}_A\left[|k\rangle\langle k'|_A \otimes |k\rangle\langle k'|_B\right]}$$

$$= \mathrm{Tr}\left[|k\rangle\langle k'|_A\right]|k\rangle\langle k'|_B$$

$$= \delta_{k,k'}\,|k\rangle\langle k'|_B$$

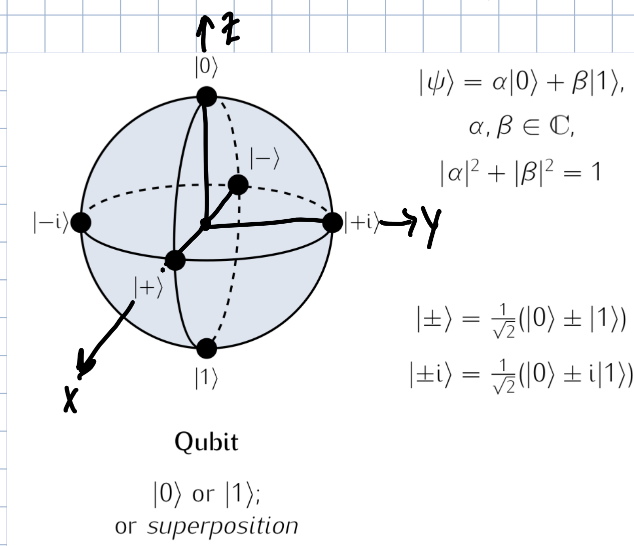
$$= \frac{1}{d}\sum_{k=0}^{d-1}|k\rangle\langle k|_B$$

$$= \frac{1}{d}\mathbb{1}_B. \blacksquare$$

(g) Mixed-state entanglement is much harder to characterize! (More on this later...).

③ Purification of mixed states

(a) Recall: For qubits, pure states are on the surface of the Bloch sphere; mixed states are inside.



General mixed states have the form $\rho = \sum_{k=1}^{N} p_k \underbrace{|\psi_k\rangle\langle\psi_k|}_{\text{pure states.}}$.

p_k: probabilities.

(b) How do mixed states arise in nature?

1. Lack of knowledge of state preparation.

$$\text{Source} \to \begin{cases} |\psi_1\rangle \text{ w/ prob. } p_1 \\ |\psi_2\rangle \text{ w/ prob. } p_2 \\ \vdots \\ |\psi_N\rangle \text{ w/ prob. } p_N. \end{cases}$$

(*) Now someone uses the source to prepare a state, but they don't tell you which one it is. How do you describe your knowledge of the system?

↳ (They don't mention the label.)

$\rho_{XS}^\dagger = \rho_{XS}$

$\rho_{XS} \geq 0$

$\mathrm{Tr}[\rho_{XS}] = 1$

$$\rho_{XS} = \sum_{k=1}^{N} p_k\, |k\rangle\langle k|_X \otimes |\psi_k\rangle\langle\psi_k|_S$$

(This is a classical-quantum state)

X: classical variable/register. ("which state is it"?)

$|k\rangle$: ONB

(*) To figure out what state the system is prepared in, we measure the "classical register" X. (w.r.t. POVM $\{|k\rangle\langle k|\}_{k=1}^{N}$)

$$\mathrm{Tr}\left[\left(|k\rangle\langle k|_X \otimes \mathbb{1}_S\right)\rho_{XS}\right] = \mathrm{Tr}\Big[\left(|k\rangle\langle k|_X \otimes \mathbb{1}_S\right)\underbrace{\sum_{k'=1}^{N} p_{k'}\, |k'\rangle\langle k'|_X \otimes |\psi_{k'}\rangle\langle\psi_{k'}|_S}_{\rho_{XS}}\Big]$$

$$= \sum_{k'=1}^{N} \mathrm{Tr}\Big[p_{k'}\, |k\rangle\underbrace{\langle k|k'\rangle}_{\delta_{k,k'}}\langle k'|_X \otimes |\psi_{k'}\rangle\langle\psi_{k'}|_S\Big]$$

$= p_k$ ✓ (as expected!)

State conditioned on outcome k) (i.e., you know what the label is).

$$\mathrm{Tr}_X\left[\frac{\left(|k\rangle\langle k|_X \otimes \mathbb{1}_S\right)\rho_{XS}\left(|k\rangle\langle k|_X \otimes \mathbb{1}_S\right)}{p_k}\right]$$

$$= \frac{1}{p_k}\sum_{k'=1}^{N} \mathrm{Tr}_X\left[\left(|k\rangle\langle k|_X \otimes \mathbb{1}_S\right)\left(p_{k'}\,|k'\rangle\langle k'|_X \otimes |\psi_{k'}\rangle\langle\psi_{k'}|_S\right)\left(|k\rangle\langle k|_X \otimes \mathbb{1}_S\right)\right]$$

$$= \frac{1}{p_k}\sum_{k'=1}^{N} p_{k'}\,\underbrace{\mathrm{Tr}_X\left[|k\rangle\langle k|k'\rangle\langle k'|k\rangle\langle k|_X \otimes |\psi_{k'}\rangle\langle\psi_{k'}|_S\right]}_{=\delta_{k',k}\,|\psi_{k'}\rangle\langle\psi_{k'}|}$$

$$= \frac{1}{p_k}\, p_k\, |\psi_k\rangle\langle\psi_k|$$

$$= |\psi_k\rangle\langle\psi_k| \checkmark \quad \text{(as expected!)}$$

⊛ But if you don't know the outcome/label? $\longrightarrow$ Trace out the register X!

$$\mathrm{Tr}_X\left(\rho_{XS}\right) = \mathrm{Tr}_X\left[\sum_{k=1}^{N} p_k\,|k\rangle\langle k|_X \otimes |\psi_k\rangle\langle\psi_k|_S\right]$$

$$= \sum_{k=1}^{N} p_k\,\underbrace{\mathrm{Tr}_X\left[|k\rangle\langle k|_X \otimes |\psi_k\rangle\langle\psi_k|_S\right]}_{=\underbrace{\mathrm{Tr}\left[|k\rangle\langle k|\right]}_{=1}|\psi_k\rangle\langle\psi_k|}$$

$$= \sum_{k=1}^{N} p_k\,|\psi_k\rangle\langle\psi_k|.$$

2. Measurement on one part of a pure state.

$$|\psi\rangle_{AB} = \sum_{k=1}^{r}\sqrt{\lambda_k}\,|e_k\rangle_A \otimes |f_k\rangle_B \longrightarrow \text{Measure } A \text{ wrt POVM } \{M_A^x\}_{x\in\mathcal{X}}$$

Conditioned on outcome x, state on B is $\frac{1}{p_x}\mathrm{Tr}_A\left[\left(\sqrt{M_A^x}\otimes\mathbb{1}_B\right)|\psi\rangle\langle\psi|_{AB}\left(\sqrt{M_A^x}\otimes\mathbb{1}_B\right)\right]$

$$\rho_x = \mathrm{Tr}\left[(M_A^x \otimes \mathbb{1}_B)|\gamma\rangle\langle\gamma|_{AB}\right] = \mathrm{Tr}\Big[M_A^x \underbrace{\mathrm{Tr}_B\left[|\gamma\rangle\langle\gamma|_{AB}\right]}\Big] = \sum_{k=1}^{r} \lambda_k \mathrm{Tr}\left[M_A^x |e_k\rangle\langle e_k|_A\right].$$

$$= \mathrm{Tr}_B\left[\sum_{k,k'=1}^{r} \sqrt{\lambda_k\lambda_{k'}}\, |e_k\rangle\langle e_{k'}|_A \otimes |f_k\rangle\langle f_{k'}|_B\right]$$

$$= \sum_{k,k'=1}^{r} \sqrt{\lambda_k\lambda_{k'}}\, |e_k\rangle\langle e_{k'}|_A\, \underbrace{\mathrm{Tr}\left[|f_k\rangle\langle f_{k'}|_B\right]}_{=\delta_{k,k'}}$$

$$= \sum_{k=1}^{r} \lambda_k |e_k\rangle\langle e_k|_A$$

$$\mathrm{Tr}_A\left[(\sqrt{M_A^x} \otimes \mathbb{1}_B)|\gamma\rangle\langle\gamma|_{AB}(\sqrt{M_A^x} \otimes \mathbb{1}_B)\right] = \mathrm{Tr}_A\left[\sum_{k,k'=1}^{r} \sqrt{M_A^x}\, |e_k\rangle\langle e_{k'}|_A \sqrt{M_A^x} \otimes |f_k\rangle\langle f_{k'}|_B\right]$$

$$= \sum_{k,k'=1}^{r} \mathrm{Tr}\left[\sqrt{M_A^x}\, |e_k\rangle\langle e_{k'}|_A \sqrt{M_A^x}\right] |f_k\rangle\langle f_{k'}|_B$$

$$= \sum_{k,k'=1}^{r} \mathrm{Tr}\left[M_A^x |e_k\rangle\langle e_{k'}|_A\right] |f_k\rangle\langle f_{k'}|_B. \rightarrow$$ Not a pure state in general.

3. Entanglement with an (unknown) environment.

Consider a mixed state $\rho_S = \sum_{k=1}^{r} \underbrace{\lambda_k}_{\text{eigenvalues}} \underbrace{|\psi_k\rangle\langle\psi_k|}_{\text{eigenvectors (orthonormal)}}$ (spectral decomposition).

⊛ Note: $\mathrm{Tr}(\rho_S) = 1 \Rightarrow \mathrm{Tr}\Big[\sum_{k=1}^{r} \lambda_k \underbrace{\mathrm{Tr}[|\psi_k\rangle\langle\psi_k|]}_{=1}\Big] = \sum_{k=1}^{r} \lambda_k = 1$ (eigenvalues sum to one).

⊛ Note: $\rho_S \geq 0 \Rightarrow \lambda_k \geq 0\ \forall k. \Rightarrow 0 \leq \lambda_k \leq 1\ \forall k.$

⊛ There exists a pure state $|\psi\rangle_{ES}$ such that $\mathrm{Tr}_E[|\psi\rangle\langle\psi|_{ES}] = \rho_S$.

$\underbrace{ES}_{\text{environment}}$

<u>Proof</u>: (by explicit construction) Consider $\sqrt{\rho_S} = \sum_{k=1}^{r} \sqrt{\lambda_k}\, |\psi_k\rangle\langle\psi_k|$

Then $\underbrace{|\psi\rangle_{ES}}_{\dim(\mathcal{H}_E) \geq r} = (\mathbb{1}_E \otimes \sqrt{\rho_S}) \underbrace{|\Gamma\rangle}_{|\Gamma\rangle = \sum_{k=0}^{d-1} |k,k\rangle} = \left(\mathbb{1}_E \otimes \sum_{k=1}^{r} \sqrt{\lambda_k}\, |\psi_k\rangle\langle\psi_k|_S\right) |\Gamma\rangle$

$\mathrm{vec}(|e_k\rangle\langle f_k|) = |\overline{f_k}\rangle \otimes |e_k\rangle$

$= \sum_{k=1}^{r} \sqrt{\lambda_k}\, \underbrace{(\mathbb{1}_E \otimes |\psi_k\rangle\langle\psi_k|_S)|k,k\rangle_{ES}}_{|\overline{\psi_k}\rangle_E \otimes |\psi_k\rangle_S} = \sum_{k=1}^{r} \sqrt{\lambda_k}\, |\overline{\psi_k}\rangle_E \otimes |\psi_k\rangle_S.$

• This is a state vector: $\langle\psi|\psi\rangle = \left(\sum_{k=1}^{r} \sqrt{\lambda_k}\, \langle\overline{\psi_k}|_E \otimes \langle\psi_k|_S\right)\left(\sum_{k'=1}^{r} \sqrt{\lambda_{k'}}\, |\overline{\psi_{k'}}\rangle_E \otimes |\psi_{k'}\rangle_S\right)$

$= \sum_{k,k'=1}^{r} \sqrt{\lambda_k \lambda_{k'}}\, \underbrace{\langle\overline{\psi_k}|\overline{\psi_{k'}}\rangle}_{\delta_{k,k'}} \underbrace{\langle\psi_k|\psi_{k'}\rangle}_{\delta_{k,k'}}$

$= \sum_{k=1}^{r} \lambda_k$

$= 1$ ✓

• $\mathrm{Tr}_E[|\psi\rangle\langle\psi|_{ES}] = \mathrm{Tr}_E[(\mathbb{1}_E \otimes \sqrt{\rho_S})|\Gamma\rangle\langle\Gamma|_{ES}(\mathbb{1}_E \otimes \sqrt{\rho_S})]$

$= \sqrt{\rho_S}\, \underbrace{\mathrm{Tr}_E[|\Gamma\rangle\langle\Gamma|_{ES}]}\, \sqrt{\rho_S}$

$|\Gamma\rangle = \sum_{k=0}^{d-1} |k,k\rangle$

$$= \sum_{k,k'=0}^{d-1} \mathrm{Tr}_E\left[|k\rangle\langle k'|_E \otimes |k\rangle\langle k'|_S \right]$$

$$= \sum_{k,k'=0}^{d-1} \delta_{k,k'} \; |k\rangle\langle k'|_S$$

$$= \sum_{k=0}^{d-1} |k\rangle\langle k|_S = \mathbb{1}_S .$$

$$= \rho_S \checkmark$$

$$\sqrt{\rho_S}\, \mathbb{1} \sqrt{\rho_S} = \rho_S$$