① Measurements

- Every orthonormal basis $\{|e_k\rangle\}_{k=1}^{d}$ in dimension d defines a measurement: operators $\{|e_k\rangle\langle e_k|\}_{k=1}^{d}$ such that $$\sum_{k=1}^{d} |e_k\rangle\langle e_k| = \mathbb{1}_d.$$

For a state ρ: $\Pr[k] = \mathrm{Tr}\left[|e_k\rangle\langle e_k|\rho\right] = \langle e_k|\rho|e_k\rangle$ ("Born rule")

$$\sum_{k=1}^{d} \Pr[k] = \mathrm{Tr}\Big[\underbrace{\Big(\sum_{k=1}^{d} |e_k\rangle\langle e_k|\Big)}_{=\mathbb{1}_d}\rho\Big] = \mathrm{Tr}[\rho] = 1.$$

⊛ Example: For the state $\rho = \frac{\mathbb{1}}{d}$ (the "maximally-mixed" state), the outcomes are completely random in every basis! (i.e., $\Pr[k] = \frac{1}{d}\ \forall k$).

- For a bipartite state $\rho_{AB} \to$ measuring A only: $\Pr[k] = \mathrm{Tr}\left[\left(|e_k\rangle\langle e_k|_A \otimes \mathbb{1}_B\right)\rho_{AB}\right]$

This is equal to $\Pr[k] = \mathrm{Tr}\Big[|e_k\rangle\langle e_k|_A \underbrace{\mathrm{Tr}_B[\rho_{AB}]}_{\text{Partial trace}}\Big]$

⊛ For an operator $M_{AB} = \sum_{i,j=0}^{d_A-1} \sum_{k,l=0}^{d_B-1} m_{i,j;k,l}\, |i\rangle\langle j|_A \otimes |k\rangle\langle l|_B$.

$\mathrm{Tr}_B[X_A \otimes Y_B] = X_A \cdot \mathrm{Tr}[Y_B]$

$$\mathrm{Tr}_B[M_{AB}] = \sum_{i,j=0}^{d_A-1} \sum_{k,l=0}^{d_B-1} m_{i,k;j,l}\, |i\rangle\langle j|_A \underbrace{\mathrm{Tr}(|k\rangle\langle l|_B)}_{\delta_{k,l}}$$

$$= \sum_{i,j=0}^{d_A-1} \left(\sum_{k=0}^{d_B-1} m_{i,k;j,k}\right) |i\rangle\langle j|_A$$

$$\mathrm{Tr}_A[M_{AB}] = \sum_{i,j=0}^{d_A-1} \sum_{k,l=0}^{d_B-1} m_{i,j;k,l}\, \underbrace{\mathrm{Tr}\left[|i\rangle\langle j|_A\right]}_{\delta_{i,j}} |k\rangle\langle l|_B = \sum_{k,l=0}^{d_B-1} \left(\sum_{i=0}^{d_A-1} m_{i,k;i,l}\right) |k\rangle\langle l|_B.$$

For a two-qubit operator: $$M_{AB} = \begin{array}{c c} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{array}$$

$$Tr_B(M_{AB}) = Tr_B \left[\begin{array}{c c} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \left(\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ \hline a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right) \end{array} \right] = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} a_{11}+a_{22} & a_{13}+a_{24} \\ a_{31}+a_{42} & a_{33}+a_{44} \end{pmatrix} \end{array}$$

$$Tr_A(M_{AB}) = Tr_A \left[\begin{array}{c c} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{array} \right] = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} a_{11}+a_{33} & a_{12}+a_{34} \\ a_{21}+a_{43} & a_{22}+a_{44} \end{pmatrix} \end{array}$$

- The most general form of a measurement is given by a <u>positive operator-valued measure (POVM)</u>: a (finite) set $\{M_x\}_{x \in \mathcal{X}}$

outcome labels.

(1) $M_x \geq 0 \;\; \forall \; x \in \mathcal{X}$

(2) $\sum_x M_x = \mathbb{1}$.

- Probabilities are given by $Pr[x] = Tr[M_x \rho]$

⊛ <u>Check</u>: $\sum_{x \in \mathcal{X}} Pr[x] = \sum_{x \in \mathcal{X}} Tr[M_x \rho] = Tr\left[\Big(\underbrace{\sum_x M_x}_{=\mathbb{1}}\Big)\rho\right] = Tr[\rho] = 1.$ ✓

- Measurements of the form $\{|e_k\rangle\langle e_k|\}_{k=1}^{d}$, for an orthonormal basis $\{|e_k\rangle\}_{k=1}^{d}$, are known as <u>projective measurements</u>, b/c $M_k \equiv |e_k\rangle\langle e_k|$ are <u>projectors</u>.

  ⊛ A <u>projector</u> is an operator/matrix satisfying $P^2 = P$.

  Check: $\left(|e_k\rangle\langle e_k|\right)^2 = |e_k\rangle\underbrace{\langle e_k|e_k\rangle}_{=1}\langle e_k| = |e_k\rangle\langle e_k|$ ✓

  - A <u>general projective measurement</u> has the form $\{\Pi_x\}_{x\in\mathcal{X}}$:

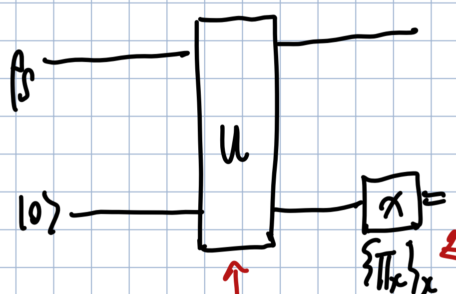
    (1) $\Pi_x \geq 0 \;\forall\, x \in \mathcal{X}$

    (2) $\Pi_x^2 = \Pi_x \;\forall\, x \in \mathcal{X}$

    (3) $\sum_{x\in\mathcal{X}} \Pi_x = \mathbb{1}$.

- <u>Theorem</u>: Every general (POVM) measurement can be written as a projective measurement on a larger system.

  <u>Proof</u>: Consider a system S; we want to measure it via a probe P.



← Projective measurement of the probe.

Unitary interaction of system and probe.

$$\rho_S \otimes |0\rangle\langle 0|_P \mapsto U(\rho_S \otimes |0\rangle\langle 0|)U^\dagger \mapsto \Pr[x] = \underbrace{\mathrm{Tr}\left[(\mathbb{1}_S \otimes \Pi_x)U(\rho_S \otimes |0\rangle\langle 0|_P)U^\dagger\right]}$$

⊛ We can write this in the form $\mathrm{Tr}[M_x \rho_S]$

$$\mathrm{Tr}\left[(\mathbb{1}_S \otimes \Pi_x) U (\rho_S \otimes |0\rangle\langle 0|_P) U^\dagger\right] = \mathrm{Tr}\left[(\rho_S \otimes |0\rangle\langle 0|) U^\dagger (\mathbb{1}_S \otimes \Pi_x) U\right]$$

$\mathrm{Tr}[ABC] = \mathrm{Tr}[BCA] = \mathrm{Tr}[CAB]$.

$(X_A \otimes Y_B) = (X_A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes Y_B)$

$\mathrm{Tr}[(\rho_A \otimes \mathbb{1}_B) M_{AB}] = \mathrm{Tr}[\rho_A \mathrm{Tr}_B[M_{AB}]]$

$$= \mathrm{Tr}\left[(\rho_S \otimes \mathbb{1}_P)(\mathbb{1}_S \otimes |0\rangle\langle 0|_P) U^\dagger (\mathbb{1}_S \otimes \Pi_x) U\right]$$

$$= \mathrm{Tr}\Big[\rho_S \underbrace{\mathrm{Tr}_P\left[(\mathbb{1}_S \otimes |0\rangle\langle 0|_P) U^\dagger (\mathbb{1}_S \otimes \Pi_x) U\right]}_{\equiv M_x.}\Big]$$

Let us show that $\{M_x\}_{x\in\mathcal{X}}$ is a POVM.

(1) Check that $M_x \geq 0$. $\langle v|M_x|v\rangle \geq 0 \ \forall\ |v\rangle$

For an arbitrary vector $|v\rangle$:

$$\langle v|M_x|v\rangle = \langle v|\,\mathrm{Tr}_P\left[(\mathbb{1}_S \otimes |0\rangle\langle 0|_P) U^\dagger (\mathbb{1}_S \otimes \Pi_x) U\right]|v\rangle$$

$$= \mathrm{Tr}\left[(|v\rangle\langle v|_S \otimes |0\rangle\langle 0|_P) U^\dagger (\mathbb{1}_S \otimes \Pi_x) U\right]$$

$\mathrm{Tr}[M|v_1\rangle\langle v_2|]$
$= \langle v_2|M|v_1\rangle$

$$= \underbrace{(\langle v|_S \otimes \langle 0|_P) U^\dagger}_{\equiv\langle v'|} (\mathbb{1}_S \otimes \Pi_x) \underbrace{U(|v\rangle_S \otimes |0\rangle_P)}_{\equiv|v'\rangle}.$$

$$= \langle v'|(\mathbb{1}_S \otimes \Pi_x)|v'\rangle$$

≥ 0 b/c $\Pi_x \geq 0$ and $\mathbb{1}_S \geq 0 \Rightarrow \mathbb{1}_S \otimes \Pi_x \geq 0$.

So $M_x \geq 0$ ✓ $\forall x$

(2) Check that $\sum_{x\in\mathcal{X}} M_x = \mathbb{1}$.

$$\sum_{x\in\mathcal{X}} M_x = \sum_{x\in\mathcal{X}} \mathrm{Tr}_P\left[(\mathbb{1}_S \otimes |0\rangle\langle 0|_P) U^\dagger (\mathbb{1}_S \otimes \Pi_x) U\right] = \mathrm{Tr}_P\Big[(\mathbb{1}_S \otimes |0\rangle\langle 0|_P) U^\dagger (\mathbb{1}_S \otimes \underbrace{\sum_{x\in\mathcal{X}} \Pi_x}_{=\mathbb{1}_P \text{ by assumption.}}) U\Big]$$

$$= \mathrm{Tr}_P\Big[(\mathbb{1}_S \otimes |0\rangle\langle 0|_P) \underbrace{U^\dagger (\mathbb{1}_S \otimes \mathbb{1}_P) U}_{\mathbb{1}_S \otimes \mathbb{1}_P}\Big]$$ $U^\dagger U = \mathbb{1}$

$$= \mathrm{Tr}_p\left[\left(\mathbb{1}_S \otimes |0\rangle\langle 0|_p\right)\left(\mathbb{1}_S \otimes \mathbb{1}_p\right)\right]$$

$$= \mathrm{Tr}_p\left[\mathbb{1}_S \otimes |0\rangle\langle 0|_p\right]$$

$$= \mathbb{1}_S. \checkmark$$

We also prove the converse: given a POVM $\{M_x\}_{x\in\mathcal{X}}$, the measurement can be realised by a unitary b/w system and probe and measurement of the probe.

Define $V = \sum_{x\in\mathcal{X}} \sqrt{M_x} \otimes \underbrace{|x\rangle}_{\text{some orthonormal vectors.}}{}_p$

(⊛ Note: M_x is Hermitian, so it has spectral decomposition $M_x = \sum_{k=1}^{r} \lambda_k |\psi_k\rangle\langle\psi_k|$. Then the square root is $\sqrt{M_x} = \sum_{k=1}^{r} \sqrt{\lambda_k} |\psi_k\rangle\langle\psi_k|$.)

⊛ V is an isometry: it satisfies $V^\dagger V = \mathbb{1}$ (but not necessarily $VV^\dagger = \mathbb{1}$.)

Check: $V^\dagger V = \left(\sum_x \sqrt{M_x} \otimes \langle x|_p\right)\left(\sum_{x'\in\mathcal{X}} \sqrt{M_{x'}} \otimes |x'\rangle_p\right)$

$$= \sum_{x,x'\in\mathcal{X}} \sqrt{M_x}\sqrt{M_{x'}} \underbrace{\langle x|x'\rangle}_{=\delta_{x,x'}} = \sum_{x\in\mathcal{X}} M_x = \mathbb{1} \checkmark$$

Now note that $\mathrm{Tr}_p\left[\left(\mathbb{1}_S \otimes |x\rangle\langle x|_p\right) VV^\dagger\right] = \mathrm{Tr}_p\left[\left(\mathbb{1}_S \otimes |x\rangle\langle x|\right)\left(\sum_{x_1\in\mathcal{X}} \sqrt{M_{x_1}} \otimes |x_1\rangle\right)\left(\sum_{x_2\in\mathcal{X}} \sqrt{M_{x_2}} \otimes \langle x_2|\right)\right]$

$$= \sum_{x_1,x_2\in\mathcal{X}} \mathrm{Tr}_p\left[\sqrt{M_{x_1}}\sqrt{M_{x_2}} \otimes |x\rangle\underbrace{\langle x|x_1\rangle}_{=\delta_{x,x_1}}\langle x_2|\right]$$

$$= \sum_{x_1,x_2\in\mathcal{X}} \sqrt{M_{x_1}}\sqrt{M_{x_2}}\, \delta_{x,x_1} \underbrace{\langle x_2|x\rangle}_{=\delta_{x,x_2}}$$

$$= M_x.$$

So for any state ρ: $\mathrm{Tr}[M_x \rho] = \mathrm{Tr}[\mathrm{Tr}_P[(\mathbb{1}_S \otimes |x\rangle\langle x|_P) VV^\dagger]\rho]$

$$= \mathrm{Tr}[(\mathbb{1}_S \otimes |x\rangle\langle x|_P) VV^\dagger (\rho_S \otimes \mathbb{1}_P)].$$

$$= \mathrm{Tr}[V^\dagger (\rho_S \otimes \mathbb{1}_P) V (\mathbb{1}_S \otimes |x\rangle\langle x|_P)].$$

This defines a projective measurement b/c $\{|x\rangle\}$ is an ONB.

⊛ <u>Fact</u>: for every isometry $V_{S\to SP}$, there exists a unitary U_{SP} such that $V_{S\to SP} = U_{SP}(\mathbb{1}_S \otimes |0\rangle_P)$.

$$= \mathrm{Tr}[U^\dagger(\rho_S \otimes |0\rangle\langle 0|_P) U (\mathbb{1}_S \otimes |x\rangle\langle x|_P)].$$

- <u>Post-measurement state after a partial measurement</u>: If we have a bipartite state ρ_{AB}, and we measure the A part, what is the resulting state on the B part?
  - Consider measurement POVM $\{M_A^{(x)}\}_{x\in\mathcal{X}}$.
  - The probabilities are $\Pr[x] = \mathrm{Tr}[(M_A^{(x)} \otimes \mathbb{1}_B)\rho_{AB}]$
  - Conditioned on the outcome x, the state of B is

$$\rho_B^{(x)} = \frac{\mathrm{Tr}_A[(M_A^{(x)} \otimes \mathbb{1}_B)\rho_{AB}]}{\underbrace{\mathrm{Tr}[(M_A^{(x)} \otimes \mathbb{1}_B)\rho_{AB}]}_{\Pr[x]}}$$

⊛ Check that $\rho_B^{(x)}$ has all of the required properties of a density matrix.

$$\mathrm{Tr}[\rho_B^{(x)}] = \frac{\mathrm{Tr}[\mathrm{Tr}_A[(M_A^{(x)} \otimes \mathbb{1}_B)\rho_{AB}]]}{\Pr[x]} = \frac{\mathrm{Tr}[(M_A^{(x)} \otimes \mathbb{1}_B)\rho_{AB}]}{\Pr[x]} = 1 \checkmark$$

② Entanglement

⊛ State vectors of two qubits belongs to the tensor-product space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

(a) – A **product state (vector)** has the form:

(State vector) $|\psi\rangle_{AB} = |\phi_1\rangle_A \otimes |\phi_2\rangle_B$, $|\phi_1\rangle \in \mathbb{C}^{d_1}$, $|\phi_2\rangle \in \mathbb{C}^{d_2}$.

(Density operator). $\rho_{AB} = \sigma_A \otimes \tau_B$, $\sigma_A \in L(\mathbb{C}^{d_A})$ and $\tau_B \in L(\mathbb{C}^{d_B})$ are density operators.

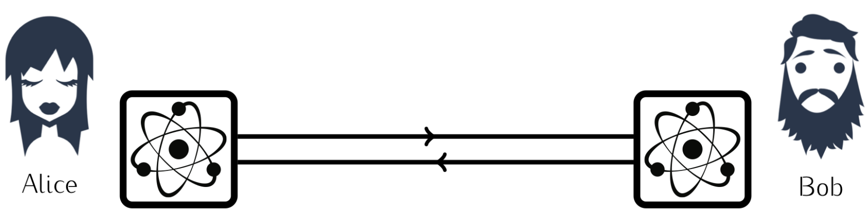
$$\rho_{AB} = \sigma_A \otimes \tau_B$$

Product state

Alice and Bob individually prepare their systems.

– A **separable state** has the form: $\rho_{AB} = \sum_x \underbrace{p(x)}_{} \sigma_A^x \otimes \tau_B^x$

↳ Also called "classically correlated" ↳ probabilities.



$$\rho_{AB} = \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \otimes \tau_B^x$$

Separable state

Alice and Bob individually prepare their systems via local operations and classical communication.

Example: $\rho_{AB} = \frac{1}{2}\left(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B\right). = \frac{1}{2}\left(|0,0\rangle\langle 0,0|_{AB} + |1,1\rangle\langle 1,1|_{AB}\right)$

This state is <u>diagonal</u> in the tensor-product basis

$\{|0\rangle_A \otimes |0\rangle_B \;\; |0\rangle_A \otimes |1\rangle_B,\; |1\rangle_A \otimes |0\rangle_B,\; |1\rangle_A \otimes |1\rangle_B\}$

$$\rho_{AB} = \frac{1}{2}\begin{array}{c} \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \\ \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \end{array} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \\ \end{array}$$

If Alice and Bob both measure in the $\{|0\rangle, |1\rangle\}$ basis, then their outcomes will be correlated!

- If Alice gets outcome "0", then so will Bob. Same with outcome "1".

$\mathrm{Tr}_A\left[\left(|0\rangle\langle 0|_A \otimes \mathbb{1}_B\right)\rho_{AB}\right] = \frac{1}{2}|0\rangle\langle 0|_B \rightarrow \rho_B^{(0)} = |0\rangle\langle 0|_B$. Also, $\rho_B^{(1)} = |1\rangle\langle 1|$.

$$= \mathrm{Tr}_A\left[\left(|0\rangle\langle 0|_A \otimes \mathbb{1}_B\right)\frac{1}{2}\left(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B\right)\right]$$

$$= \frac{1}{2}\left(\mathrm{Tr}_A\left[\left(|0\rangle\langle 0|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B\right)\right] + \mathrm{Tr}_A\left[\left(\cancelto{0}{|0\rangle\langle 0|1\rangle\langle 1|_A} \otimes |1\rangle\langle 1|_B\right)\right]\right)$$

$$= \frac{1}{2}\mathrm{Tr}_A\left[|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B\right]$$

$$= \frac{1}{2}|0\rangle\langle 0|_B$$

- <u>But this is not the case for all measurements!</u>

If they both measure in $\{|+\rangle, |-\rangle\}$ basis instead, then the outcomes are completely uncorrelated!

$\Pr[+,+] = \Pr[+,-] = \Pr[-,+] = \Pr[-,-] = \frac{1}{4}$.

If Alice gets outcome "+" (happens with probability $\frac{1}{2}$), then Bob's state is $\rho_B^{(+)} = \frac{1}{1/2} \mathrm{Tr}_A[(|+\rangle\langle+|_A \otimes \mathbb{1}_B)\rho_{AB}] = \frac{\mathbb{1}}{2}$ → completely random for Bob!

- So, in this case, the correlations depend on the basis choice!

⊛ <u>Aside</u>: All states of two probabilistic bits can be expressed as diagonal density matrices in the computational basis.

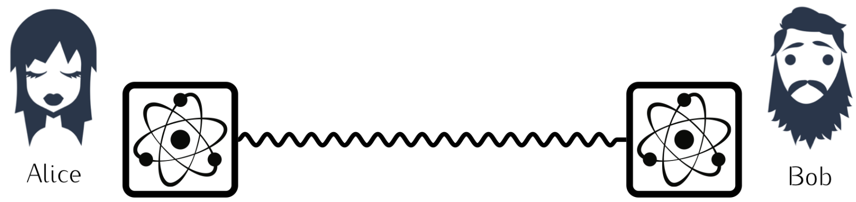
$$\rho_{AB} = \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{pmatrix} p_{00} & & & 0 \\ & p_{01} & & \\ & & p_{10} & \\ 0 & & & p_{11} \end{pmatrix}$$
$$\begin{array}{cccc} 00 & 01 & 10 & 11 \end{array}$$

(⊛ Even one bit!

$\rho = \begin{pmatrix} p_1 & \\ & p_2 \end{pmatrix} = \frac{1}{2}(p_1|0\rangle\langle 0| + p_2|1\rangle\langle 1|)$

describes an arbitrary state of one bit!)

- An **entangled state** is NOT a separable state.



$$\rho_{AB} \neq \sum_{x \in \mathcal{X}} p(x)\, \sigma_A^x \otimes \tau_B^x$$

Entangled state

Correlations between Alice and Bob are non-local.
State of the individual systems not sufficient to describe the pair.

$Z|0\rangle = |0\rangle$
$Z|1\rangle = -|1\rangle$

(b) Examples:

$|\Phi_{0,0}\rangle = |\Phi^+\rangle$, $(Z \otimes \mathbb{1})|\Phi^+\rangle = \frac{1}{\sqrt{2}}(Z|0\rangle|0\rangle + Z|1\rangle|1\rangle)$

$= \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$ $\quad = \frac{1}{\sqrt{2}}(|0,0\rangle - |1,1\rangle)$

$= |\Phi^-\rangle$

1. The **Bell states**

$$\Phi^{\pm} = |\Phi^\pm\rangle\langle\Phi^\pm|, \quad |\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle \pm |1,1\rangle)$$

$$\Psi^{\pm} = |\Psi^\pm\rangle\langle\Psi^\pm|, \quad |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle \pm |1,0\rangle).$$

Observe that $\{|\Phi^\pm\rangle, |\Psi^\pm\rangle\} \longleftrightarrow \{\underbrace{(Z^z X^x \otimes \mathbb{1})|\Phi^+\rangle}_{|\Phi_{z,x}\rangle} : z, x \in \{0,1\}\}$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $Z^0 = \mathbb{1}$, $Z^1 = Z$
$X^0 = \mathbb{1}$, $X^1 = X$

$0,0 \leftrightarrow \Phi^+$
$0,1 \leftrightarrow \Psi^+$
$1,0 \leftrightarrow \Phi^-$
$1,1 \leftrightarrow \Psi^-$

In d dimensions: $|\Phi_d^+\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k,k\rangle$ $\quad (d=3)\ \frac{1}{\sqrt{3}}(|0\rangle|0\rangle + |1\rangle|1\rangle + |2\rangle|2\rangle)$

(See later for the other d-dimensional Bell states.)

$$|\Phi^+\rangle = \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \quad |\Phi^-\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad |\Psi^+\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad |\Psi^-\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$$(X \otimes \mathbb{1})|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(\underbrace{X|0\rangle}_{|1\rangle}|0\rangle + \underbrace{X|1\rangle}_{|0\rangle}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(|1\rangle|0\rangle + |0\rangle|1\rangle\right) = |\Psi^+\rangle$$

$$(ZX \otimes \mathbb{1})|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(\underbrace{ZX|0\rangle}_{\substack{Z|1\rangle \\ = -|1\rangle}}|0\rangle + \underbrace{ZX|1\rangle}_{\substack{Z|0\rangle \\ = |0\rangle}}|1\rangle\right) = \frac{1}{\sqrt{2}}\left(-|1\rangle|0\rangle + |0\rangle|1\rangle\right) = |\Psi^-\rangle$$

$\circledast$ $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$, $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \alpha_1\alpha_2|0,0\rangle + \alpha_1\beta_2|0,1\rangle + \beta_1\alpha_2|1,0\rangle + \beta_1\beta_2|1,1\rangle \neq \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$$

There does not exist values of $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $|\psi_1\rangle \otimes |\psi_2\rangle = |\Phi^+\rangle$.

$\circledast$ $\{|\Phi_{z,x}\rangle : z,x \in \{0,1\}\}$ is an ONB for $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$\underline{\underline{U}}|z,x\rangle = |\Phi_{z,x}\rangle$$

↳ U is a unitary matrix : $U^\dagger U = \mathbb{1} = UU^\dagger \Leftrightarrow U^\dagger = U^{-1}$
(change-of-basis matrix).

$\Rightarrow \{|\Phi_{z,x}\rangle\langle\Phi_{z',x'}| : z,x,z',x' \in \{0,1\}\}$ is an ONB for $L(\mathbb{C}^2 \otimes \mathbb{C}^2)$

$$\Rightarrow \rho_{AB} = \sum_{\substack{z,x \in \{0,1\} \\ z',x' \in \{0,1\}}} \underbrace{C_{\substack{z,x \\ z',x'}}}_{\in \mathbb{C}} |\Phi_{z,x}\rangle\langle\Phi_{z',x'}|_{AB}$$

$\circledast$ $\sum_{z,x \in \{0,1\}} \Phi_{AB}^{z,x} = \mathbb{1}_{AB} \longrightarrow$ Bell states form a measurement (POVM)

↳ Important ingredient in teleportation.