① Recap: Measurements

- To extract classical information from a qubit, we must **measure** it
- Recall: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow$ Probability of 0: $|\alpha|^2$; Probability of 1: $|\beta|^2$ } ⊛ Axiom of quantum mechanics! (aka "Born Rule").

⊛ Note: $|\alpha|^2 = |\langle 0|\psi\rangle|^2$, $|\beta|^2 = |\langle 1|\psi\rangle|^2$

Also: $\underbrace{|\langle 0|\psi\rangle|^2}_{\equiv \Pr[0]} = \langle 0|\psi\rangle\langle\psi|0\rangle = \mathrm{Tr}[|0\rangle\langle 0|\psi\rangle\langle\psi|]$

"Measurement operators"

$\underbrace{|\langle 1|\psi\rangle|^2}_{\equiv \Pr[1]} = \langle 1|\psi\rangle\langle\psi|1\rangle = \mathrm{Tr}[|1\rangle\langle 1|\psi\rangle\langle\psi|]$.

$\Pr[0] + \Pr[1] = \mathrm{Tr}[(\overbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}^{\mathbb{1}})\rho] = \mathrm{Tr}[\rho] = 1$

- For a density matrix ρ: $\Pr[0] = \mathrm{Tr}[|0\rangle\langle 0|\rho]$, $\Pr[1] = \mathrm{Tr}[|1\rangle\langle 1|\rho]$.

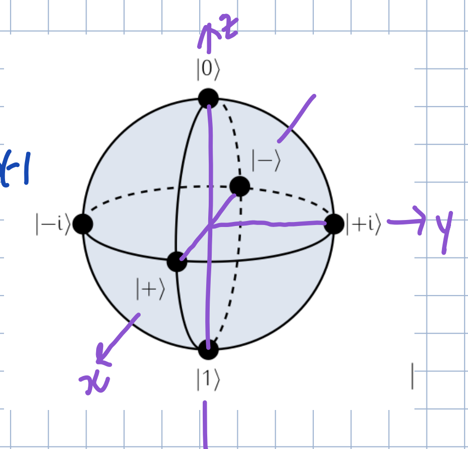
⊛ This is often called a "computational-basis measurement" or a "$\{|0\rangle, |1\rangle\}$-basis measurement".

⊛ Recall that $\{|0\rangle, |1\rangle\}$ is also the eigenvectors of Pauli-Z:

$Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$. $\rightarrow$ So we also sometimes say "Pauli-Z measurement"

⊛ Circuit symbol: $|\psi\rangle$ —$\boxed{x}$=

- Pauli-X measurement: measure along x-axis; equivalent to measuring in basis $\{|+\rangle, |-\rangle\} \rightarrow |+\rangle\langle+| + |-\rangle\langle-| = \mathbb{1}$



⊛ Recall: $|+\rangle = H|0\rangle$, $|-\rangle = H|1\rangle$, $H \equiv$ Hadamard gate $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

⊛ H unitary $\Rightarrow \{|+\rangle, |-\rangle\}$ is a basis!

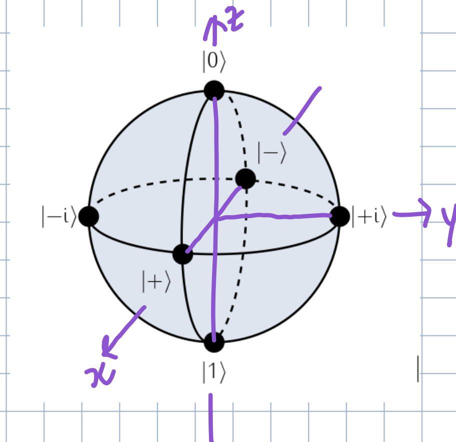
⊛ For a state vector $|\psi\rangle$: $\Pr[+] = |\langle +|\psi\rangle|^2 = \langle +|\psi\rangle\langle\psi|+\rangle = \mathrm{Tr}[|+\rangle\langle +|\psi\rangle\langle\psi|]$.

$\Pr[-] = |\langle -|\psi\rangle|^2 = \langle -|\psi\rangle\langle\psi|-\rangle = \mathrm{Tr}[|-\rangle\langle -|\psi\rangle\langle\psi|]$

⊛ For a density operator ρ: $\Pr[+] = \mathrm{Tr}[\underbrace{|+\rangle\langle +|}\rho]$, $\Pr[-] = \mathrm{Tr}[\underbrace{|-\rangle\langle -|}\rho]$.

Measurement operators.

⊛ Circuit Symbol: $|\psi\rangle$ —[H]—[x]=



- Pauli-Y measurement: measure along y-axis; equivalent to measuring in basis $\{|{+i}\rangle, |{-i}\rangle\}$

⊛ Recall: $|{+i}\rangle = SH|0\rangle$, $|{-i}\rangle = SH|1\rangle$, $H \equiv$ Hadamard gate $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$. $S \equiv$ phase gate $= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

⊛ SH is unitary $\Rightarrow \{|{+i}\rangle, |{-i}\rangle\}$ is a basis!

⊛ For a state vector $|\psi\rangle$: $\Pr[+i] = |\langle +i|\psi\rangle|^2 = \langle +i|\psi\rangle\langle\psi|+i\rangle = \mathrm{Tr}[|{+i}\rangle\langle +i|\psi\rangle\langle\psi|]$.

$|{+i}\rangle\langle +i| + |{-i}\rangle\langle -i| = \mathbb{1}$. $\Pr[-i] = |\langle -i|\psi\rangle|^2 = \langle -i|\psi\rangle\langle\psi|-i\rangle = \mathrm{Tr}[|{-i}\rangle\langle -i|\psi\rangle\langle\psi|]$

⊛ For a density operator ρ: $\Pr[+i] = \mathrm{Tr}[\underbrace{|{+i}\rangle\langle +i|}\rho]$, $\Pr[-i] = \mathrm{Tr}[\underbrace{|{-i}\rangle\langle -i|}\rho]$.

Measurement operators.

- Measuring multiple qubits.

  - Consider state vector $|\psi\rangle$ of n qubits $(|\psi\rangle \in (\mathbb{C}^2)^{\otimes n})$. → or density matrix ρ.

  - Computational-basis measurement is a $\{|0\rangle, |1\rangle\}$ measurement on each qubit

  - Outcome probabilities: $\Pr[0,0,1] = |\overbrace{\langle 0,0,1|}^{\langle 0|\otimes\langle 0|\otimes\langle 1|}\psi\rangle|^2$ (for three qubits).

$\Pr[x_1,x_2,x_3] = |\langle x_1,x_2,x_3|\psi\rangle|^2$, $x_1,x_2,x_3 \in \{0,1\}$.

$|x_1,x_2,x_3\rangle\langle x_1,x_2,x_3| = |x_1\rangle\langle x_1| \otimes |x_2\rangle\langle x_2| \otimes |x_3\rangle\langle x_3|$

$= \mathrm{Tr}[|x_1,x_2,x_3\rangle\langle x_1,x_2,x_3|\psi\rangle\langle\psi|]$

⊛ For a density operator ρ: $\mathrm{Tr}[|x_1,x_2,x_3\rangle\langle x_1,x_2,x_3|\rho]$.

- Expectation Value of an Observable.

⊛ Axiom of Quantum Mechanics: Observables are described mathematically by Hermitian operators. $H^\dagger = H$.

- Recall: Hermitian operators have a spectral/eigenvalue decomposition:

$$H|\psi_k\rangle = \lambda_k|\psi_k\rangle$$

$$H = \sum_{k=1}^{d} \underbrace{\lambda_k}_{\text{eigenvalues (real numbers).}} \underbrace{|\psi_k\rangle\langle\psi_k|}_{\text{eigenvectors.}}$$

⊛ Examples: the Pauli operators!

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = |+\rangle\langle+| - |-\rangle\langle-|$$

$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = |+i\rangle\langle+i| - |-i\rangle\langle-i|$$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |0\rangle\langle 0| - |1\rangle\langle 1|.$$

- Eigenvectors are orthonormal basis

$\Rightarrow$ they define a measurement!

$$\sum_{k=1}^{d} |\psi_k\rangle\langle\psi_k| = \mathbb{1}.$$

For a density operator ρ:

$\Pr[k] = \mathrm{Tr}(|\psi_k\rangle\langle\psi_k|\rho) \rightarrow$ The associated "observed" value is λ_k.

(Example: $k \equiv$ energy level of a molecule, $\lambda_k \equiv$ the energy value).

$$\sum_{k=1}^{d} \Pr(k) = \sum_{k=1}^{d} \mathrm{Tr}[|\psi_k\rangle\langle\psi_k|\rho] = \mathrm{Tr}\Big[\big(\underbrace{\sum_{k=1}^{d} |\psi_k\rangle\langle\psi_k|}_{=\mathbb{1}}\big)\rho\Big] = \mathrm{Tr}[\rho] = 1.$$

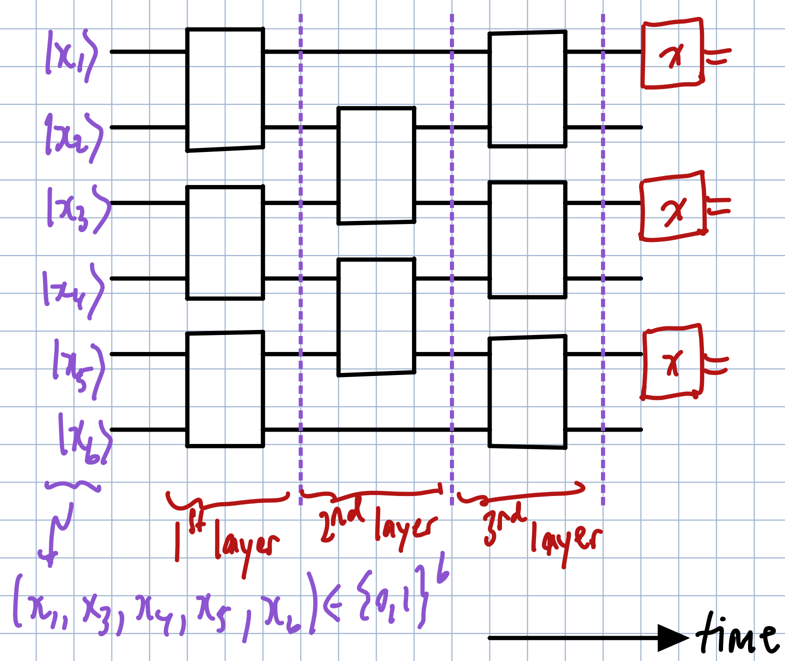
- The expected (aka "mean" or "average") value of an observable, given a state ρ, is

$$\sum_{k=1}^{d} \Pr(k)\cdot\lambda_k = \sum_{k=1}^{d} \mathrm{Tr}[|\psi_k\rangle\langle\psi_k|\rho]\cdot\lambda_k$$

$$= \mathrm{Tr}\Big[\big(\underbrace{\sum_{k=1}^{d} \lambda_k|\psi_k\rangle\langle\psi_k|}_{H}\big)\rho\Big] = \underline{\underline{\mathrm{Tr}[H\rho]}}$$

(⊛ Recall: for a random variable X, $\mathbb{E}[X] = \sum_x \Pr[X=x]\cdot x$)

② Partial Measurements: Measuring Only Some Qubits



⊛ Suppose we only measure the 1^{st}, 3^{rd}, and 5^{th} qubits.

- To get the measurement outcome probabilities, we apply the measurement operators only on the relevant qubits.

⊛ Notation: States and operators for multiple qubits and qudits.

- For two qubits, the standard basis is $\{\underbrace{\underset{1^{st}\text{ qubit}}{|0\rangle} \otimes \underset{2^{nd}\text{ qubit.}}{|0\rangle}}_{\equiv|0,0\rangle}, \underbrace{|0\rangle \otimes |1\rangle}_{\equiv|0,1\rangle}, \underbrace{|1\rangle \otimes |0\rangle}_{\equiv|1,0\rangle}, \underbrace{|1\rangle \otimes |1\rangle}_{\equiv|1,1\rangle}\}$

  We often label the qubits by A, B, C, ...

  So we write $|0,0\rangle_{AB} \equiv \underset{\text{"qubit A"}}{|0\rangle_A} \otimes \underset{\text{"qubit B"}}{|0\rangle_B}$. → For state vector $|\psi\rangle_{AB} \in \underset{\text{qubit A}}{\mathbb{C}^2} \otimes \underset{\text{qubit B.}}{\mathbb{C}^2}$

  $|\psi\rangle_{AB} = a|0,0\rangle_{AB} + b|0,1\rangle_{AB} + c|1,0\rangle_{AB} + d|1,1\rangle_{AB}$.

- Similar for three or more qubits: $|\psi\rangle_{ABC} = \sum_{\underline{x} \in \{0,1\}^3} c_{\underline{x}} \underbrace{|\underline{x}\rangle_{ABC}}_{\equiv |x_1, x_2, x_3\rangle_{ABC}.}$

For n qubits: $|\psi\rangle_{A_1 A_2 \cdots A_n} = \sum_{\vec{x} \in \{0,1\}^n} c_{\vec{x}} \underbrace{|\vec{x}\rangle_{A_1 A_2 \cdots A_n}}$

$\equiv |x_1, x_2, \ldots, x_n\rangle_{A_1 A_2 \cdots A_n}$

$\equiv |x_1\rangle_{A_1} \otimes |x_2\rangle_{A_2} \otimes \cdots \otimes |x_n\rangle_{A_n}$

- We do the same for linear operators. (including density operators)

$M \in L(\mathbb{C}^d)$ $\quad M_{AB} \in L(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}) \rightarrow (d_A \cdot d_B) \times (d_A \cdot d_B)$ matrix

System A has dimension d_A; B has dimension d_B.

$$M_{AB} = \sum_{i,j=0}^{d_A - 1} \sum_{k,l=0}^{d_B - 1} m_{i,k;j,l} \underbrace{|i,k\rangle\langle j,l|_{AB}}$$

$\equiv |i\rangle\langle j|_A \otimes |k\rangle\langle l|_B$. $\rightarrow$ These form a basis for $L(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$.

$$M_{A_1 A_2 \cdots A_n} \in L(\mathbb{C}^{d_{A_1}} \otimes \mathbb{C}^{d_{A_2}} \otimes \cdots \otimes \mathbb{C}^{d_{A_n}}) \rightarrow (d_{A_1} \cdot d_{A_2} \cdots d_{A_n}) \times (d_{A_1} \cdot d_{A_2} \cdots d_{A_n}) \text{ matrix.}$$

- Suppose we have a two-qubit density operator ρ_{AB}, and we measure the qubit A in the computational basis.

Measurement operators are $|0\rangle\langle 0|$ and $|1\rangle\langle 1|$.

Probabilities are: $\Pr[0] = \mathrm{Tr}[(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)\rho_{AB}]$

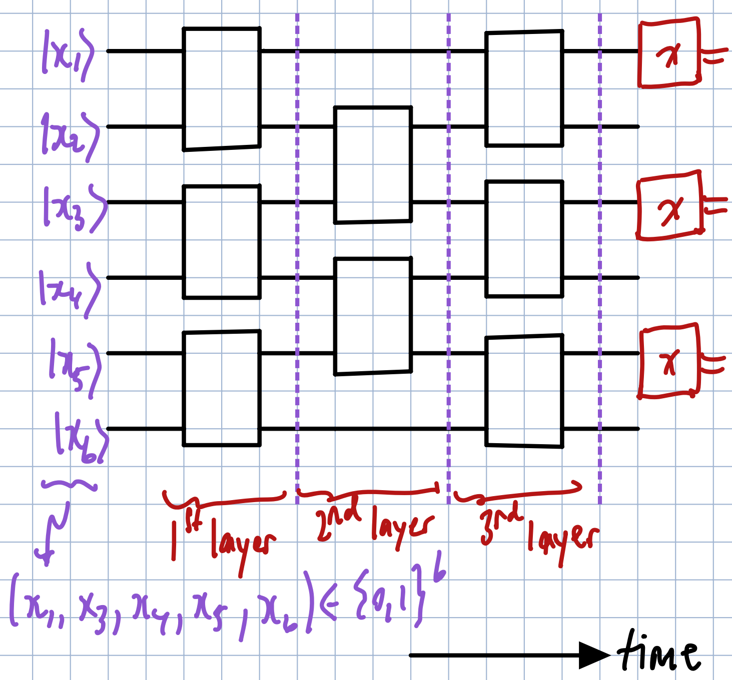
$\Pr[1] = \mathrm{Tr}[\underbrace{(|1\rangle\langle 1|_A \otimes \mathbb{1}_B)}\rho_{AB}]$

We measure qubit A only, so we apply the measurement operator only on A; we apply identity ("do nothing") on B.

If we measure only B, then: $\Pr[x] = \mathrm{Tr}[(\mathbb{1}_A \otimes |x\rangle\langle x|_B)\rho_{AB}]$, $x \in \{0,1\}$.

If we measure <u>both</u> A + B, then: $\Pr[x_1,x_2] = \mathrm{Tr}[(\underbrace{|x_1\rangle\langle x_1|_A \otimes |x_2\rangle\langle x_2|_B}_{|x_1,x_2\rangle\langle x_1,x_2|_{AB}})\rho_{AB}]$.

- Suppose we only measure the 1st, 3rd, and 5th qubits.



- Let the state before measurement be given by the density operator $\rho_{A_1A_2\cdots A_6}$

- The probability distribution is given by:

$$\Pr[x_1,x_2,x_3] = \mathrm{Tr}[(|x_1\rangle\langle x_1|_{A_1} \otimes \mathbb{1}_{A_2} \otimes |x_2\rangle\langle x_2|_{A_3} \otimes \mathbb{1}_{A_4} \otimes |x_3\rangle\langle x_3|_{A_5} \otimes \mathbb{1}_{A_6})\rho_{A_1\cdots A_6}].$$

$x_1, x_2, x_3 \in \{0,1\}$.

③ The Partial Trace

- When we only measure parts of a quantum system, we only need to know the state of that <u>subsystem</u> — we can "ignore" the rest of the system

- The partial trace describes how (mathematically) to describe ignoring/discarding parts of a system.

- For a linear operator $M_{AB} \in L(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$, the (full) trace is:

$$\mathrm{Tr}[M_{AB}] = \sum_{k=0}^{d_A-1} \sum_{l=0}^{d_B-1} \langle k,l|_{AB}\, M_{AB}\, |k,l\rangle_{AB}.$$

(sum of the diagonal elements).

(Recall that $\{|k,l\rangle : k \in \{0,1,\dots,d_A-1\},\ l \in \{0,1,\dots,d_B-1\}\}$ is a basis.)

$\mathrm{Tr}[M] = \sum_{k=0}^{d-1} \langle k|M|k\rangle$

$$\mathrm{Tr}[M_{AB}] = \sum_{k=0}^{d_A-1} \sum_{l=0}^{d_B-1} (\langle k|_A \otimes \langle l|_B)\, M_{AB}\, (|k\rangle_A \otimes |l\rangle_B).$$

- Partial trace over B :

$$\mathrm{Tr}_B[M_{AB}] = \cancel{\sum_{k=0}^{d_A-1}} \sum_{l=0}^{d_B-1} (\overset{\mathbb{1}_A}{\cancel{\langle k|_A}} \otimes \langle l|_B)\, M_{AB}\, (\overset{\mathbb{1}_A}{\cancel{|k\rangle_A}} \otimes |l\rangle_B)$$

$$= \underline{\underline{\sum_{l=0}^{d_B-1} (\mathbb{1}_A \otimes \langle l|_B)\, M_{AB}\, (\mathbb{1}_A \otimes |l\rangle_B)}}.$$

⊛ Note : the outcome is an operator! Not a scalar, like the full trace.

- Partial trace over A :

$$\mathrm{Tr}_A[M_{AB}] = \sum_{k=0}^{d_A-1} \cancel{\sum_{l=0}^{d_B-1}} (\langle k|_A \otimes \overset{\mathbb{1}_B}{\cancel{\langle l|_B}})\, M_{AB}\, (|k\rangle_A \otimes \overset{\mathbb{1}_B}{\cancel{|l\rangle_B}})$$

$$= \underline{\underline{\sum_{k=0}^{d_A-1} (\langle k|_A \otimes \mathbb{1}_B)\, M_{AB}\, (|k\rangle_A \otimes \mathbb{1}_B)}}.$$

⊛ Note : the outcome is an operator! Not a scalar, like the full trace.

- Some Properties:

(1)
$$\mathrm{Tr}_A[M_A \otimes N_B] = \sum_{k=0}^{d_A-1} \underbrace{(\langle k|_A \otimes \mathbb{1}_B)(M_A \otimes N_B)}_{= (\langle k|_A M_A \otimes \mathbb{1}_B N_B) = \langle k|_A M_A \otimes N_B}(|k\rangle_A \otimes \mathbb{1}_B)$$

$$= \sum_{k=0}^{d_A-1} \langle k|_A M_A |k\rangle_A\, N_B$$

$$= \mathrm{Tr}[M_A]\, N_B.$$

⊛ Property of tensor product: $(M_1 \otimes M_2)(N_1 \otimes N_2) = M_1 N_1 \otimes M_2 N_2$

$(\langle k|_A M_A \otimes N_B)(|k\rangle_A \otimes \mathbb{1}_B) = \langle k|M_A|k\rangle \cdot N_B.$

(2) $\mathrm{Tr}_B[M_A \otimes N_B] = \mathrm{Tr}[N_B]\, M_A.$

$$M_{AB} \in L(\mathbb{C}^2 \otimes \mathbb{C}^2) \rightarrow \quad M_{AB} = \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \end{array}$$

$$= \sum_{i,j=0}^{1} \sum_{k,l=0}^{1} M_{i,k;j,l} \, |i\rangle\langle j|_A \otimes |k\rangle\langle l|_B .$$

$$\mathrm{Tr}_B[M_{AB}] = \sum_{i,j=0}^{1} \sum_{k,l=0}^{1} M_{i,k;j,l} \, \underbrace{\mathrm{Tr}_B\left[|i\rangle\langle j|_A \otimes |k\rangle\langle l|_B\right]}_{= |i\rangle\langle j| \cdot \underbrace{\mathrm{Tr}[|k\rangle\langle l|]}_{\delta_{k,l}}} = \sum_{i,j=0}^{1} \sum_{k=0}^{1} M_{i,k;j,k} \, |i\rangle\langle j|_A$$

$$\mathrm{Tr}_B[M_{AB}] = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} a_{11}+a_{22} & a_{13}+a_{24} \\ a_{31}+a_{42} & a_{33}+a_{44} \end{pmatrix} \end{array}$$

- Recall: for a computational-basis measurement on qubit A of two-qudit system AB in state ρ_{AB}: $\Pr[x] = \mathrm{Tr}\left[\left(|x\rangle\langle x|_A \otimes \mathbb{1}_B\right)\rho_{AB}\right]$ $= \mathrm{Tr}\left[|x\rangle\langle x|_A \mathrm{Tr}_B(\rho_{AB})\right].$

$\{|k,\ell\rangle : k \in \{0,1,\dots,d_A-1\},\ \ell \in \{0,1,\dots,d_B-1\}\}$ is a basis

$$\Rightarrow \mathrm{Tr}\left[\left(|x\rangle\langle x|_A \otimes \mathbb{1}_B\right)\rho_{AB}\right] = \sum_{k=0}^{d_A-1}\sum_{\ell=0}^{d_B-1} \langle k,\ell|_{AB}\left(\left(|x\rangle\langle x|_A \otimes \mathbb{1}_B\right)\rho_{AB}\right)|k,\ell\rangle_{AB}.$$

(This is a sum over the diagonal elements.)

$$= \sum_{k=0}^{d_A-1}\sum_{\ell=0}^{d_B-1} \left(\langle k|_A \otimes \langle \ell|_B\right)\left(\left(|x\rangle\langle x|_A \otimes \mathbb{1}_B\right)\rho_{AB}\right)\left(|k\rangle_A \otimes |\ell\rangle_B\right).$$

(⊛ Property of tensor product: $(M_1 \otimes M_2)(N_1 \otimes N_2) = M_1N_1 \otimes M_2N_2$)

$$= \langle k|_A \left(\mathbb{1}_A \otimes \langle \ell|_B\right)\left(|x\rangle\langle x|_A \otimes \mathbb{1}_B\right)\rho_{AB}\left(\mathbb{1}_A \otimes |\ell\rangle_B\right)|k\rangle_A$$

(⊛ $(\mathbb{1}_A \otimes M_B)(N_A \otimes \mathbb{1}_B) = N_A \otimes M_B = (N_A \otimes \mathbb{1}_B)(\mathbb{1}_A \otimes M_B)$.)

$$= \langle k|_A |x\rangle\langle x|_A \left(\mathbb{1}_A \otimes \langle \ell|_B\right)\rho_{AB}\left(\mathbb{1}_A \otimes |\ell\rangle_B\right)|k\rangle_A$$

$$= \sum_{k=0}^{d_A-1}\sum_{\ell=0}^{d_B-1} \langle k|_A |x\rangle\langle x|_A \left(\mathbb{1}_A \otimes \langle \ell|_B\right)\rho_{AB}\left(\mathbb{1}_A \otimes |\ell\rangle_B\right)|k\rangle_A$$

$$= \sum_{k=0}^{d_A-1} \langle k|_A |x\rangle\langle x|_A \underbrace{\left(\sum_{\ell=0}^{d_B-1}\left(\mathbb{1}_A \otimes \langle \ell|_B\right)\rho_{AB}\left(\mathbb{1}_A \otimes |\ell\rangle_B\right)\right)}_{\equiv \mathrm{Tr}_B[\rho_{AB}]}|k\rangle_A$$

$\equiv \mathrm{Tr}_B[\rho_{AB}] \rightarrow$ the partial trace!

Compare with the full trace:

$$\mathrm{Tr}[\rho_{AB}] = \sum_{k=0}^{d_A-1} \sum_{l=0}^{d_B-1} \left(\langle k|_A \otimes \langle l|_B\right) \rho_{AB} \left(|k\rangle_A \otimes |l\rangle_B\right).$$

$$= \mathrm{Tr}\Big[|x\rangle\langle x|_A \underbrace{\mathrm{Tr}_B[\rho_{AB}]}_{\equiv \rho_A} \Big]$$

For measurement of B instead: $\Pr(x) = \mathrm{Tr}\left[\left(\mathbb{1}_A \otimes |x\rangle\langle x|_B \right) \rho_{AB} \right] = \mathrm{Tr}\Big[|x\rangle\langle x|_B \underbrace{\mathrm{Tr}_A[\rho_{AB}]}_{\equiv \rho_B .} \Big].$