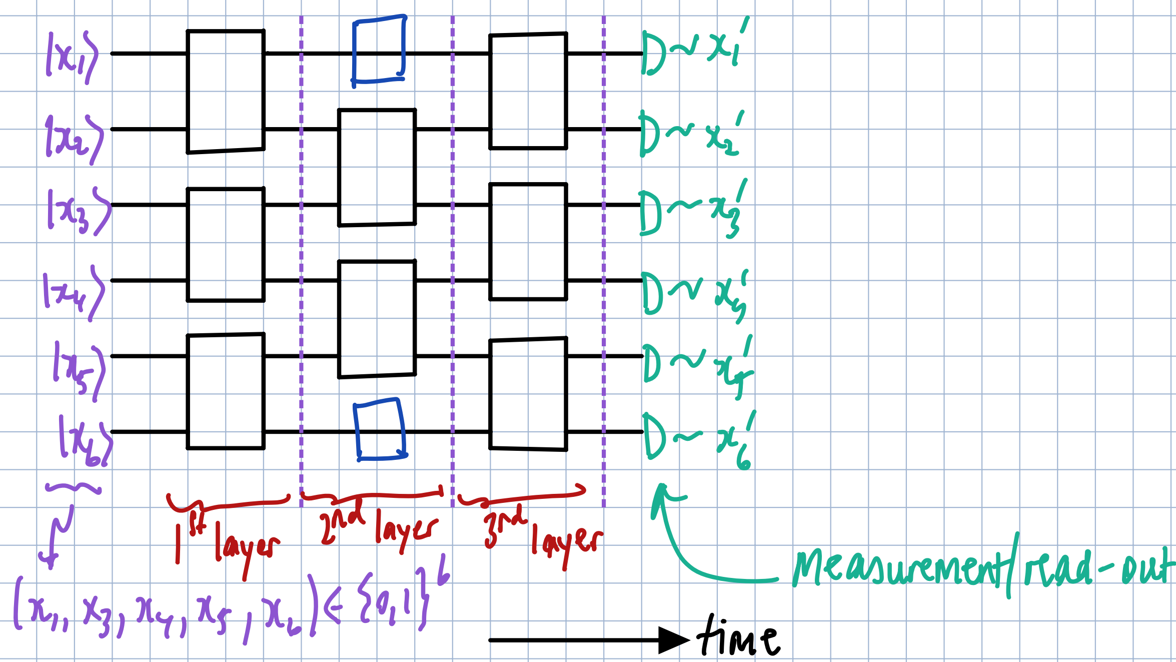
① Recap: Quantum Gates

- Quantum gates describe how the states of qubits evolve — Similar to classical logic gates like AND, OR, NOT.

- Composing gates leads to <u>circuits</u>, which are used to describe quantum computations.



- Mathematically, quantum gates are described by <u>unitary matrices/operators</u> $U^\dagger U = UU^\dagger = \mathbb{1}$. They are <u>norm and trace preserving</u>. They describe <u>basis changes</u>.

$|\psi\rangle \to U|\psi\rangle \to \| U|\psi\rangle \| = \| |\psi\rangle \|$ $\quad \{U|k\rangle\}_{k=0}^{d-1}$

$\rho \to U\rho U^\dagger \to \mathrm{Tr}[U\rho U^\dagger] = \mathrm{Tr}[\rho]$.

- <u>Examples</u>:

  - Pauli gates: $-\boxed{X}- \;\to\; X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (X|0\rangle = |1\rangle,\ X|1\rangle = |0\rangle)$

    $-\boxed{Y}- \;\to\; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad (Y|0\rangle = i|1\rangle,\ Y|1\rangle = -i|0\rangle)$

    $-\boxed{Z}- \;\to\; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (Z|0\rangle = |0\rangle,\ Z|1\rangle = -|1\rangle)$

- **Hadamard gate**: —[H]— $\rightarrow H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

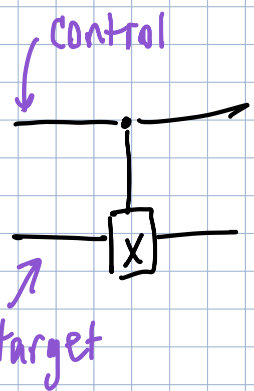
$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$
$$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right).$$

- **Phase gate**: —[S]— $\rightarrow S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

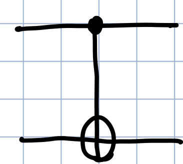
- **T-gate**: —[T]— $\rightarrow T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

- **CNOT / CX gate**:

"controlled-X"



or



$$CNOT = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$|0\rangle|0\rangle \longmapsto |0\rangle|0\rangle$

$|0\rangle|1\rangle \longmapsto |0\rangle|1\rangle$

$|1\rangle|0\rangle \longmapsto |1\rangle X|0\rangle = |1\rangle|1\rangle$

$|1\rangle|1\rangle \longmapsto |1\rangle X|1\rangle = |1\rangle|0\rangle$

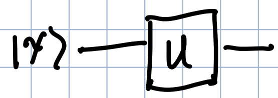
- **Rotation Gates**:

$R_X(\theta) = e^{-i\frac{\theta}{2}X} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)X$ $\rightarrow$ rotation around X-axis by angle θ.

$R_Y(\theta) = e^{-i\frac{\theta}{2}Y} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)Y$ $\rightarrow$ rotation around Y-axis by angle θ.

$R_Z(\theta) = e^{-i\frac{\theta}{2}Z} = \cos\left(\frac{\theta}{2}\right)\mathbb{1} - i\sin\left(\frac{\theta}{2}\right)Z$ $\rightarrow$ rotation around Z-axis by angle θ.

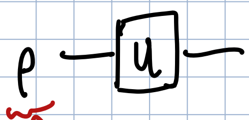
② Translating b/w circuit diagram and math.



$\leftrightarrow \quad U|\psi\rangle$

$\leftrightarrow \quad U_2 U_1 |\psi\rangle$

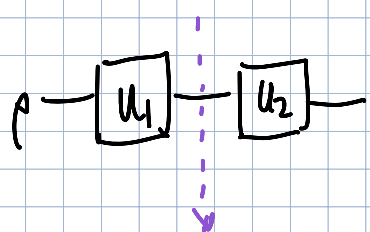
state vector (representing pure state).



$\leftrightarrow \quad U \rho U^\dagger$

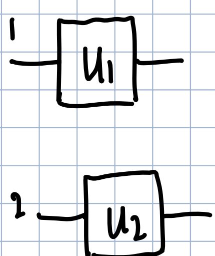
$\rho = |\psi\rangle\langle\psi| \to U|\psi\rangle\langle\psi|U^\dagger$

density operator (representing mixed state).



$U_1 \rho U_1^\dagger$

$\leftrightarrow \quad U_2 U_1 \rho U_1^\dagger U_2^\dagger$



$\leftrightarrow \quad U_1 \otimes U_2.$

tensor product of matrices.

U_1 acts on qubit 1

U_2 acts on qubit 2.

Recall: $|v_1\rangle = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$, $|v_2\rangle = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \Rightarrow |\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} a_1 a_2 \\ a_1 b_2 \\ b_1 a_2 \\ b_1 b_2 \end{pmatrix} = \begin{pmatrix} a_1 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \\ a_2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \end{pmatrix}$

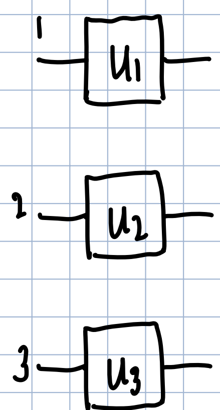
$$M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \Rightarrow M_1 \otimes M_2 = \begin{pmatrix} a_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} & b_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ c_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} & d_1 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} a_1 a_2 & a_1 b_2 & b_1 a_2 & b_1 b_2 \\ a_1 c_2 & a_1 d_2 & b_1 c_2 & b_1 d_2 \\ c_1 a_2 & c_1 b_2 & d_1 a_2 & d_1 b_2 \\ c_1 c_2 & c_1 d_2 & d_1 c_2 & d_1 d_2 \end{pmatrix}$$

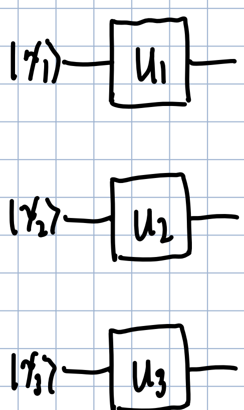
$Tr[M \otimes N] = Tr[N]\,Tr[M]$.

$\langle M_1 \otimes M_2, N_1 \otimes N_2 \rangle$
$= \langle M_1, N_1 \rangle \langle M_2, N_2 \rangle$

$\langle M, N \rangle = Tr[M^\dagger N]$.



$\longleftrightarrow \quad U_1 \otimes U_2 \otimes U_3$.



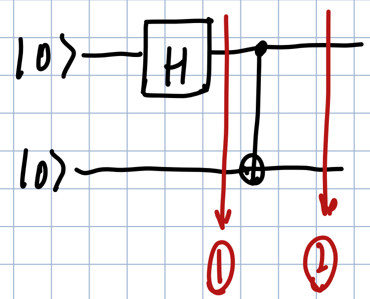
$\longleftrightarrow \quad (U_1 \otimes U_2 \otimes U_3)(|\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle)$

$= U_1|\psi_1\rangle \otimes U_2|\psi_2\rangle \otimes U_3|\psi_3\rangle$.

• **Tracking the State of qubits through a circuit**

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$

$|+\rangle|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|0\rangle$

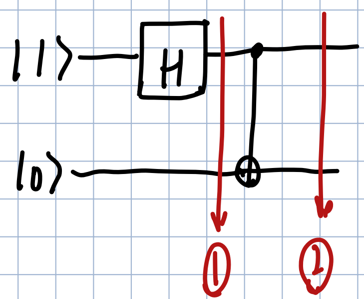


$$|0\rangle|0\rangle \overset{1}{\mapsto} H|0\rangle|0\rangle = |+\rangle|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle+|1\rangle|0\rangle)$$

$$\overset{2}{\mapsto} \frac{1}{\sqrt{2}}\Big(\underbrace{CNOT|0\rangle|0\rangle}_{|0\rangle|0\rangle} + \underbrace{CNOT|1\rangle|0\rangle}_{|1\rangle|1\rangle}\Big)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle+|1\rangle|1\rangle)$$

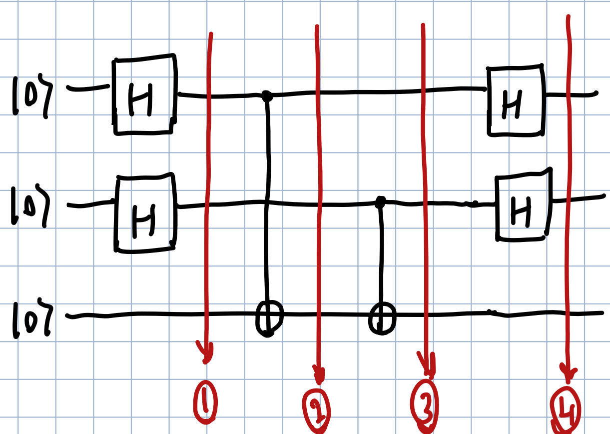
$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$



$$|1\rangle|0\rangle \overset{1}{\mapsto} H|1\rangle|0\rangle = |-\rangle|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle-|1\rangle|0\rangle)$$

$$\overset{2}{\mapsto} \frac{1}{\sqrt{2}}(CNOT|0\rangle|0\rangle - CNOT|1\rangle|0\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle-|1\rangle|1\rangle).$$



$$|0\rangle|0\rangle|0\rangle \overset{1}{\mapsto} |+\rangle|+\rangle|0\rangle$$

$$= \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|0\rangle$$

$$= \frac{1}{2}(|0\rangle|0\rangle|0\rangle + |0\rangle|1\rangle|0\rangle + |1\rangle|0\rangle|0\rangle + |1\rangle|1\rangle|0\rangle)$$

$$\overset{2}{\mapsto} \frac{1}{2}(|0\rangle|0\rangle|0\rangle + |0\rangle|1\rangle|0\rangle + |1\rangle|0\rangle|1\rangle + |1\rangle|1\rangle|1\rangle)$$

$$\overset{3}{\mapsto} \frac{1}{2}(|0\rangle|0\rangle|0\rangle + |0\rangle|1\rangle|1\rangle + |1\rangle|0\rangle|1\rangle + |1\rangle|1\rangle|0\rangle)$$

$$\overset{4}{\mapsto} \frac{1}{2}(|+\rangle|+\rangle|0\rangle + |+\rangle|-\rangle|1\rangle + |-\rangle|+\rangle|1\rangle + |-\rangle|-\rangle|0\rangle)$$

$$= \frac{1}{2}\Big(\underbrace{(|+\rangle|+\rangle + |-\rangle|-\rangle)}_{(a)}|0\rangle + \underbrace{(|+\rangle|-\rangle + |-\rangle|+\rangle)}_{(b)}|1\rangle\Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\underbrace{\frac{1}{\sqrt{2}}(|0\rangle|0\rangle+|1\rangle|1\rangle)}|0\rangle + \underbrace{\frac{1}{\sqrt{2}}(|0\rangle|0\rangle-|1\rangle|1\rangle)}|1\rangle\Big)$$

(a) $\frac{1}{2}(|0\rangle|0\rangle + \cancel{|0\rangle|1\rangle} + \cancel{|1\rangle|0\rangle} + |1\rangle|1\rangle + |0\rangle|0\rangle - \cancel{|0\rangle|1\rangle} - \cancel{|1\rangle|0\rangle} + |1\rangle|1\rangle)$

$= |0\rangle|0\rangle + |1\rangle|1\rangle$

(b) $\frac{1}{2}(|0\rangle|0\rangle - \cancel{|0\rangle|1\rangle} + \cancel{|1\rangle|0\rangle} - |1\rangle|1\rangle + |0\rangle|0\rangle + \cancel{|0\rangle|1\rangle} - \cancel{|1\rangle|0\rangle} - |1\rangle|1\rangle)$

$= |0\rangle|0\rangle - |1\rangle|1\rangle$

③ Measurements

- To extract classical information from a qubit, we must **measure** it

- Recall: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow$ Probability of 0 : $|\alpha|^2$
  Probability of 1 : $|\beta|^2$ ⊛ Axiom of quantum mechanics! (aka "Born Rule").

  $\alpha = \langle 0|\psi\rangle$, $\beta = \langle 1|\psi\rangle$

⊛ Note: $|\alpha|^2 = |\langle 0|\psi\rangle|^2$, $|\beta|^2 = |\langle 1|\psi\rangle|^2$

Also: $\underbrace{|\langle 0|\psi\rangle|^2}_{\equiv Pr[0]} = \langle 0|\psi\rangle\langle\psi|0\rangle = Tr[|0\rangle\langle 0|\psi\rangle\langle\psi|]$

$\langle v_1|M|v_2\rangle = Tr[M|v_2\rangle\langle v_1|]$.

$\underbrace{|\langle 1|\psi\rangle|^2}_{\equiv Pr[1]} = \langle 1|\psi\rangle\langle\psi|1\rangle = Tr[|1\rangle\langle 1|\psi\rangle\langle\psi|]$.

Check: $Pr[0] + Pr[1] = Tr[\underbrace{|0\rangle\langle 0|\psi\rangle\langle\psi|}_{M_1}] + Tr[\underbrace{|1\rangle\langle 1|\psi\rangle\langle\psi|}_{M_2}]$

$Tr[M_1 + M_2] = Tr[M_1] + Tr[M_2]$

$Tr[|0\rangle\langle 0|\psi\rangle\langle\psi| + |1\rangle\langle 1|\psi\rangle\langle\psi|]$

$= Tr[(\underbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}_{= \mathbb{1}})|\psi\rangle\langle\psi|]$

$= Tr[|\psi\rangle\langle\psi|]$

$= \langle\psi|\psi\rangle$

$= 1$ ✓

⊛ This is often called a "computational-basis measurement" or a "$\{|0\rangle, |1\rangle\}$-basis measurement".

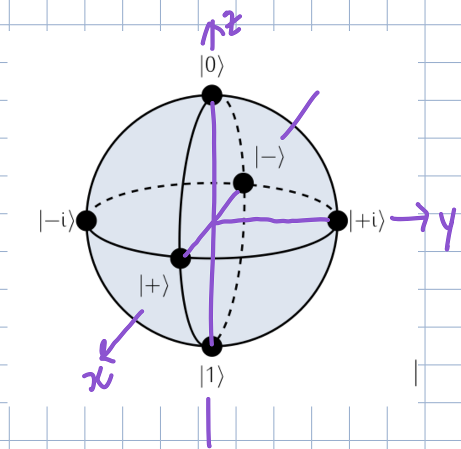
⊛ Recall that $\{|0\rangle, |1\rangle\}$ is also the eigenvectors of Pauli-Z:

$Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$. $\rightarrow$ So we also sometimes say "**Pauli-Z measurement**"

⊛ Circuit symbol:

⊛ **Observe**: If $|\psi\rangle = |0\rangle$, then $\Pr[0] = 1$, $\Pr[1] = 0$.
If $|\psi\rangle = |1\rangle$, then $\Pr[0] = 0$, $\Pr[1] = 1$.

- We can measure with respect to other Pauli operators too!



- **Pauli-X measurement**: measure along x-axis; equivalent to measuring in basis $\{|+\rangle, |-\rangle\}$

⊛ Recall: $|+\rangle = H|0\rangle$, $|-\rangle = H|1\rangle$, $H \equiv$ Hadamard gate $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

⊛ H unitary $\Rightarrow \{|+\rangle, |-\rangle\}$ is a basis!

⊛ For a state vector $|\psi\rangle$:

$$\Pr[+] = |\langle +|\psi\rangle|^2 = \langle +|\psi\rangle\langle\psi|+\rangle = \mathrm{Tr}[|+\rangle\langle +|\psi\rangle\langle\psi|].$$

$$\Pr[-] = |\langle -|\psi\rangle|^2 = \langle -|\psi\rangle\langle\psi|-\rangle = \mathrm{Tr}[|-\rangle\langle -|\psi\rangle\langle\psi|]$$

$X|+\rangle = |+\rangle$, $X|-\rangle = -|-\rangle$

Check: $\Pr[+] + \Pr[-] = \mathrm{Tr}[|+\rangle\langle +|\psi\rangle\langle\psi|] + \mathrm{Tr}[|-\rangle\langle -|\psi\rangle\langle\psi|]$

$$= \mathrm{Tr}[\underbrace{(|+\rangle\langle +| + |-\rangle\langle -|)}\,|\psi\rangle\langle\psi|] = \mathrm{Tr}[|\psi\rangle\langle\psi|] = 1 \quad ✓$$

$|+\rangle = H|0\rangle$
$|-\rangle = H|1\rangle$

$$= H|0\rangle\langle 0|H^\dagger + H|1\rangle\langle 1|H^\dagger = H|0\rangle\langle 0|H + H|1\rangle\langle 1|H$$

$$= H\,(\underbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}_{=\mathbb{1}})\,H$$

$$= HH = \mathbb{1}$$

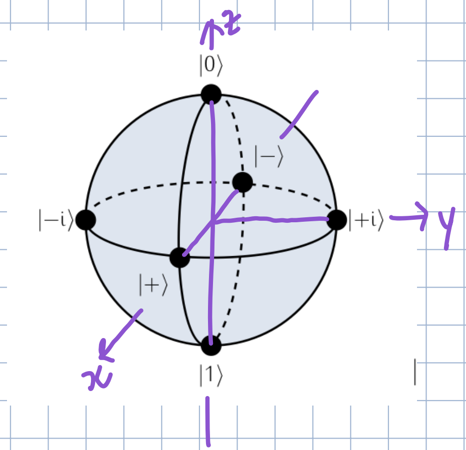
⊛ **Observe**: $\Pr[+] = \mathrm{Tr}[|+\rangle\langle +|\psi\rangle\langle\psi|] = \mathrm{Tr}[H|0\rangle\langle 0|H|\psi\rangle\langle\psi|] = \mathrm{Tr}[|0\rangle\langle 0|\overbrace{H|\psi\rangle}^{|\psi'\rangle}\langle\psi|H]$

$\Pr[-] = \mathrm{Tr}[|-\rangle\langle -|\psi\rangle\langle\psi|] = \mathrm{Tr}[H|1\rangle\langle 1|H|\psi\rangle\langle\psi|] = \mathrm{Tr}[|1\rangle\langle 1|H|\psi\rangle\langle\psi|H]$.

$\Rightarrow$ Apply H gate to $|\psi\rangle$, then measure in $\{|0\rangle, |1\rangle\}$ basis!

⊛ Circuit Symbol: $|\psi\rangle$ — $\boxed{H}$ — $\boxed{x}$ =

$|\pm i\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm i|1\rangle\right)$



- **Pauli-Y measurement**: measure along y-axis; equivalent to measuring in basis $\{|+i\rangle, |-i\rangle\}$

⊛ Recall: $|+i\rangle = SH|0\rangle$, $|-i\rangle = SH|1\rangle$, $H \equiv$ Hadamard gate $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.

⊛ SH unitary $\Rightarrow \{|+i\rangle, |-i\rangle\}$ is a basis!

$S \equiv$ phase gate $= \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

⊛ For a state vector $|\psi\rangle$:

$$\Pr[+i] = |\langle +i|\psi\rangle|^2 = \langle +i|\psi\rangle\langle\psi|+i\rangle = \mathrm{Tr}\left[|+i\rangle\langle +i|\psi\rangle\langle\psi|\right].$$

$$\Pr[-i] = |\langle -i|\psi\rangle|^2 = \langle -i|\psi\rangle\langle\psi|-i\rangle = \mathrm{Tr}\left[|-i\rangle\langle -i|\psi\rangle\langle\psi|\right]$$

Check: $\Pr[+i] + \Pr[-i] = \mathrm{Tr}\left[|+i\rangle\langle +i|\psi\rangle\langle\psi|\right] + \mathrm{Tr}\left[|-i\rangle\langle -i|\psi\rangle\langle\psi|\right]$

$$= \mathrm{Tr}\left[\left(|+i\rangle\langle +i| + |-i\rangle\langle -i|\right)|\psi\rangle\langle\psi|\right] = \mathrm{Tr}\left[|\psi\rangle\langle\psi|\right] = 1 \checkmark$$

$|+i\rangle = SH|0\rangle$

$|-i\rangle = SH|1\rangle$.

$$|+i\rangle\langle +i| + |-i\rangle\langle -i| = SH|0\rangle\langle 0|HS^\dagger + SH|1\rangle\langle 1|HS^\dagger = SH|0\rangle\langle 0|HS^\dagger + SH|1\rangle\langle 1|HS^\dagger$$

$$= SH\underbrace{\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)}_{=\mathbb{1}}HS^\dagger$$

$$= S\underbrace{H \cdot H}_{=\mathbb{1}}S^\dagger = SS^\dagger = \mathbb{1}$$

⊛ Observe: $\Pr[+i] = \mathrm{Tr}\left[|+i\rangle\langle +i|\psi\rangle\langle\psi|\right] = \mathrm{Tr}\left[SH|0\rangle\langle 0|HS^\dagger|\psi\rangle\langle\psi|\right] = \mathrm{Tr}\left[|0\rangle\langle 0|\underbrace{HS^\dagger|\psi\rangle}_{|\psi'\rangle}\langle\psi|SH\right]$

$\Pr[-i] = \mathrm{Tr}\left[|-i\rangle\langle -i|\psi\rangle\langle\psi|\right] = \mathrm{Tr}\left[SH|1\rangle\langle 1|HS^\dagger|\psi\rangle\langle\psi|\right] = \mathrm{Tr}\left[|1\rangle\langle 1|HS^\dagger|\psi\rangle\langle\psi|SH\right]$

$\Rightarrow$ Apply $S^\dagger$, then H gate to $|\psi\rangle$, then measure in $\{|0\rangle, |1\rangle\}$ basis!

⊛ Circuit Symbol: $|\psi\rangle$ — $\boxed{S^\dagger}$ — $\boxed{H}$ — $\boxed{x}$ =

- Measuring multiple qubits.

- Consider state vector $|\psi\rangle$ of n qubits $\left(|\psi\rangle \in (\mathbb{C}^2)^{\otimes n} \right)$.
- Computational-basis measurement is a $\{|0\rangle, |1\rangle\}$ measurement on each qubit
- Outcome probabilities: $\Pr[0,0,1] = |\overbrace{\langle 0,0,1|}^{\langle 0| \otimes \langle 0| \otimes \langle 1|}\psi\rangle|^2$ (for three qubits).

$$\Pr[x_1, x_2, x_3] = |\langle x_1, x_2, x_3|\psi\rangle|^2, \quad x_1, x_2, x_3 \in \{0,1\}.$$

- What is the probability that the first qubit is 0?

$$\Pr[1^{st} \text{ qubit } 0] = \Pr[0,0,0] + \Pr[0,0,1] + \Pr[0,1,0] + \Pr[0,1,1].$$

000	100
001	101
010	110
011	111

$$\Pr[2^{nd} \text{ qubit } 1] = \Pr[0,1,0] + \Pr[0,1,1] + \Pr[1,1,0] + \Pr[1,1,1].$$

- We can simultaneously measure each qubit in a different basis.

Example: measure 1^{st} + 3^{rd} qubit in Z-basis, 2^{nd} in X-basis.

$$\Pr[0,+,1] = |\underbrace{\langle 0,+,1|}_{= \langle 0| \otimes \langle +| \otimes \langle 1|}\psi\rangle|^2$$

④ Observables

- In quantum mechanics, measurement outcomes are not deterministic — they occur with some probability.
- What is the expected outcome of a measurement?
- Recall: For a probability mass function $p_X(x)$, $\left(\begin{array}{l} p_X(x) \in [0,1] \ \forall x \in \mathcal{X} \\ \sum_{x \in \mathcal{X}} p_X(x) = 1 \end{array} \right)$

  Random variable → $\Pr[X = x] = p_X(x)$.

  the expected value is

$$\mathbb{E}[X] = \sum_x p_X(x) \cdot x$$

- We can measure wrt any orthonormal basis — consider basis $\{|e_k\rangle\}_{k=1}^{d}$.

  Outcomes are labeled by k

  Outcome k is associated with value λ_k. ↔ random variable X

  For a state vector $|\psi\rangle \in \mathbb{C}^d$, the probability distribution is

$$\Pr(k) = |\langle e_k|\psi\rangle|^2 \equiv \Pr[X = \lambda_k]$$

$\langle v_1|M|v_2\rangle = \mathrm{Tr}[M|v_2\rangle\langle v_1|]$.

The expected value of X is:

$$\mathbb{E}[X] = \sum_{k=1}^{d} \Pr[X = \lambda_k] \cdot \lambda_k = \sum_{k=1}^{d} |\langle e_k|\psi\rangle|^2 \cdot \lambda_k = \sum_{k=1}^{d} \langle e_k|\psi\rangle\langle\psi|e_k\rangle \cdot \lambda_k$$

$$= \sum_{k=1}^{d} \mathrm{Tr}(|e_k\rangle\langle e_k|\psi\rangle\langle\psi|) \cdot \lambda_k = \mathrm{Tr}\Big[\underbrace{\Big(\sum_{k=1}^{d} \lambda_k |e_k\rangle\langle e_k|\Big)}_{H}|\psi\rangle\langle\psi|\Big] = \mathrm{Tr}[H|\psi\rangle\langle\psi|].$$

H → Hermitian operator.

$$H|e_{k'}\rangle = \sum_{k=1}^{d} \lambda_k |e_k\rangle\underbrace{\langle e_k|e_{k'}\rangle}_{\delta_{k,k'}} = \lambda_{k'}|e_{k'}\rangle$$