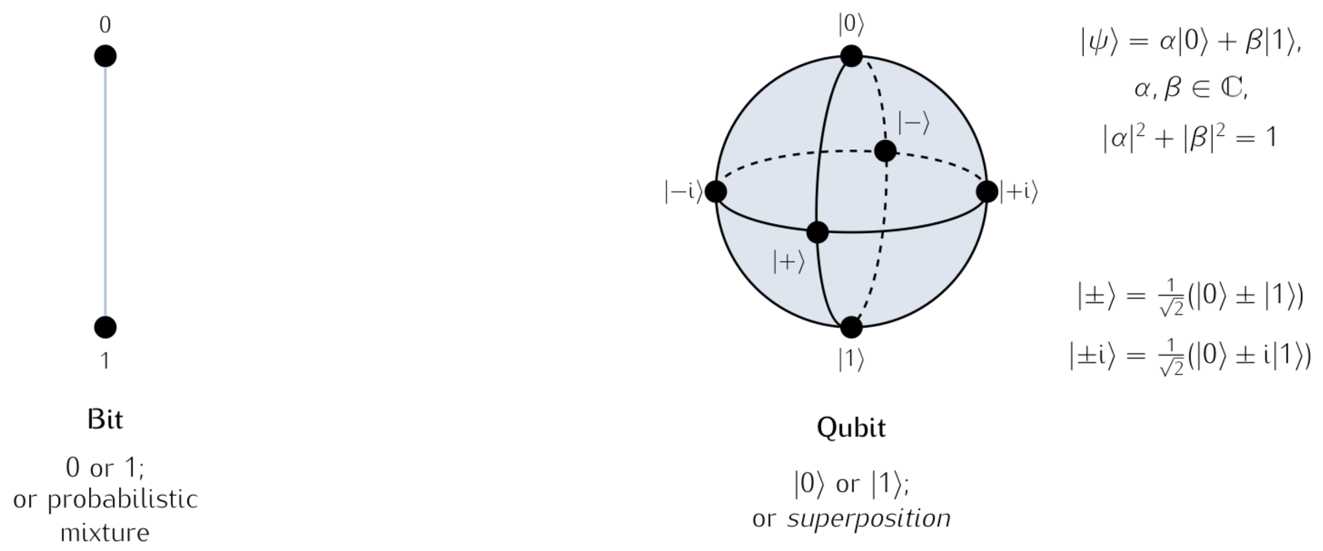
① Quantum States



⊛ States of live on the surface of the <u>Bloch sphere</u>. — they are described by 2-dimensional <u>state vectors</u>.

$\{|k\rangle\}_{k=0}^{d-1}$ $\{|e_k\rangle\}_{k=1}^{d}$

$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1.$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \quad |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(States <u>inside</u> the sphere are also possible — they are described by <u>density matrices</u>. We will see this today!)

② Matrices (aka "linear operators")

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |0\rangle\langle 0| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

↗ column indices.

- $M = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}, \quad a, b, c, d \in \mathbb{C}. \quad \rightarrow \quad M = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|$

  ↗ row indices

$a = \langle 0|M|0\rangle, \quad b = \langle 0|M|1\rangle, \quad c = \langle 1|M|0\rangle,$
$d = \langle 1|M|1\rangle.$

- <u>General d×d matrix</u>: $M = \sum_{i,j=0}^{d-1} m_{ij} |i\rangle\langle j|, \quad m_{ij} = \langle i|M|j\rangle$

  → matrix elements.

✱ **Notation**: $L(\mathbb{C}^d) \equiv$ set of all matrices/linear operators acting on $\mathbb{C}^d$ (i.e., $d\times d$ matrices) (dimension is d^2.)

- For any vector $|v\rangle \in \mathbb{C}^d$, $|v\rangle\langle v| \in L(\mathbb{C}^d)$ is a $d\times d$ matrix.

$$|v\rangle = \sum_{k=0}^{d-1} v_k |k\rangle \Rightarrow |v\rangle\langle v| = \left(\sum_{k=0}^{d-1} v_k |k\rangle\right)\left(\sum_{k'=0}^{d-1} \bar{v}_{k'} \langle k'|\right) = \sum_{k,k'=0}^{d-1} v_k \bar{v}_{k'} |k\rangle\langle k'|.$$

- Identity matrix: $\mathbb{1} \rightarrow \mathbb{1}|v\rangle = |v\rangle$ for all $|v\rangle \in \mathbb{C}^d$.

$M_{ij} = \delta_{ij}$

$$\mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & \ddots & 1 \end{pmatrix} = \sum_{k=0}^{d-1} |k\rangle\langle k| \rightarrow \text{For any orthonormal basis } \{|e_k\rangle\}_{k=1}^{d} : \mathbb{1} = \sum_{k=1}^{d} |e_k\rangle\langle e_k|.$$

- Trace of a matrix: sum of the diagonal elements.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \mathrm{Tr}[M] = a + d = \langle 0|M|0\rangle + \langle 1|M|1\rangle.$$

In general: $M = \sum_{i,j=0}^{d-1} M_{ij} |i\rangle\langle j| \Rightarrow \mathrm{Tr}[M] = \sum_{i=0}^{d-1} M_{i,i} = \sum_{i=0}^{d-1} \langle i|M|i\rangle.$

✱ Consider a state vector $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \Rightarrow \langle\psi| = \bar{\alpha}\langle 0| + \bar{\beta}\langle 1|$, $|\alpha|^2 + |\beta|^2 = 1$.

$$\Rightarrow |\psi\rangle\langle\psi| = \left(\alpha|0\rangle + \beta|1\rangle\right)\left(\bar{\alpha}\langle 0| + \bar{\beta}\langle 1|\right) = \alpha\cdot\bar{\alpha}|0\rangle\langle 0| + \alpha\bar{\beta}|0\rangle\langle 1| + \beta\bar{\alpha}|1\rangle\langle 0| + \beta\bar{\beta}|1\rangle\langle 1|$$

$$= |\alpha|^2 |0\rangle\langle 0| + \alpha\bar{\beta}|0\rangle\langle 1| + \beta\bar{\alpha}|1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1| = \begin{pmatrix} |\alpha|^2 & \alpha\bar{\beta} \\ \beta\bar{\alpha} & |\beta|^2 \end{pmatrix}$$

$|\alpha|^2 = \Pr[0]$

$|\beta|^2 = \Pr[1]$

Now take the trace: $\mathrm{Tr}[|\psi\rangle\langle\psi|] = |\alpha|^2 + |\beta|^2 = 1$.

In general: $\mathrm{Tr}[|v\rangle\langle v|] = \langle v|v\rangle$ for all $|v\rangle \in \mathbb{C}^d$.

Proof: Just use the definition!

$$\mathrm{Tr}\big[|v\rangle\langle v|\big] = \sum_{k=0}^{d-1} \langle k|v\rangle\langle v|k\rangle = \sum_{k=0}^{d-1} \langle v|k\rangle\langle k|v\rangle = \langle v|\underbrace{\left(\sum_{k=0}^{d-1} |k\rangle\langle k|\right)}_{=\mathbb{1}}|v\rangle = \langle v|v\rangle. \ \blacksquare$$

$$\langle v|\underbrace{\mathbb{1}|v\rangle}_{|v\rangle}$$

Similarly: $\mathrm{Tr}\big[M|v_1\rangle\langle v_2|\big] = \langle v_2|M|v_1\rangle$ for all $|v_1\rangle, |v_2\rangle \in \mathbb{C}^d$, $M \in L(\mathbb{C}^d)$.

Proof: $$\mathrm{Tr}\big[M|v_1\rangle\langle v_2|\big] = \sum_{k=0}^{d-1} \langle k|M|v_1\rangle\langle v_2|k\rangle = \sum_{k=0}^{d-1} \langle v_2|k\rangle\langle k|M|v_1\rangle = \langle v_2|\underbrace{\left(\sum_{k=0}^{d-1} |k\rangle\langle k|\right)}_{=\mathbb{1}} M|v_1\rangle$$

$$= \langle v_2|\underbrace{\mathbb{1}\cdot M}_{=M}|v_1\rangle$$

$$= \langle v_2|M|v_1\rangle. \ \blacksquare$$

$\mathbb{1}|v\rangle = |v\rangle$, $\mathbb{1}\cdot M = M$

- Transpose of a matrix: flip the rows and columns.

• $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow M^T = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$

Note: $\big(|i\rangle\langle j|\big)^T = |j\rangle\langle i|$.

$(M_1 + M_2)^T = M_1^T + M_2^T$

• In general: $M = \sum_{i,j=0}^{d-1} m_{ij} |i\rangle\langle j| \Rightarrow M^T = \sum_{i,j=0}^{d-1} m_{ij} |j\rangle\langle i|$

$$M^T = \left(\sum_{i,j=0}^{d-1} m_{ij} |i\rangle\langle j|\right)^T = \sum_{i,j=0}^{d-1} m_{ij} \big(|i\rangle\langle j|\big)^T$$

A matrix M is called symmetric if $M^T = M$.

$M^T = M \rightarrow m_{ij} = m_{ji}$

- **Conjugate Transpose of a Matrix:**

- Recall for vectors: $|v\rangle = \begin{pmatrix} v_0 \\ v_1 \\ \vdots \\ v_{d-1} \end{pmatrix} \Rightarrow \langle v| = \begin{pmatrix} \bar{v}_0 & \bar{v}_1 & \cdots & \bar{v}_{d-1} \end{pmatrix}$

  OR: $|v\rangle = \sum_{k=0}^{d-1} v_k |k\rangle \Rightarrow \langle v| = \sum_{k=0}^{d-1} \bar{v}_k \langle k|$.

- Similar for matrices! $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow M^{\dagger} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$. "dagger"

- In general: $M = \sum_{i,j=0}^{d-1} m_{i,j} |i\rangle\langle j| \Rightarrow M^{\dagger} = \sum_{i,j=0}^{d-1} \overline{m_{i,j}} |j\rangle\langle i|$

⊛ <u>Note</u>: $\left(|i\rangle\langle j|\right)^{\dagger} = |j\rangle\langle i|$.

⊛ A matrix M is called <u>Hermitian</u> if $M^{\dagger} = M$. $\rightarrow m_{i,j} = \overline{m_{j,i}}$

$M = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$, $\bar{a} = a$, $\bar{d} = d$ $(a, d \in \mathbb{R})$

- **Inner Product and Orthonormal Bases of Matrices.**

- Recall vector inner product: $|u\rangle = \sum_{k=0}^{d-1} u_k |k\rangle$, $|v\rangle = \sum_{k=0}^{d-1} v_k |k\rangle$

  $\Rightarrow \langle u|v\rangle = \left(\sum_{k=0}^{d-1} \bar{u}_k \langle k|\right)\left(\sum_{k=0}^{d-1} v_k |k\rangle\right) = \sum_{k=0}^{d-1} \bar{u}_k v_k$

- Matrix inner product: $\langle M_1, M_2 \rangle = \mathrm{Tr}[M_1^{\dagger} M_2]$. , $\langle M_2, M_1 \rangle = \overline{\langle M_1, M_2 \rangle}$

  From this, we define a <u>norm for matrices</u>: $\|M\|_2 = \sqrt{\langle M, M \rangle} = \sqrt{\mathrm{Tr}[M^{\dagger} M]}$

- Matrices M_1, M_2 are <u>orthogonal</u> if $\langle M_1, M_2 \rangle = 0$.

- We have bases for matrices as well! $\underbrace{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}}$

Example: $d=2$, recall $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\underbrace{|0\rangle\langle 0|}_{\begin{pmatrix}1&0\\0&0\end{pmatrix}} + b\underbrace{|0\rangle\langle 1|}_{\begin{pmatrix}0&1\\0&0\end{pmatrix}} + c\underbrace{|1\rangle\langle 0|}_{\begin{pmatrix}0&0\\1&0\end{pmatrix}} + d\underbrace{|1\rangle\langle 1|}_{\begin{pmatrix}0&0\\0&1\end{pmatrix}}$.

The set $\underbrace{\{|0\rangle\langle 0|, |0\rangle\langle 1|, |1\rangle\langle 0|, |1\rangle\langle 1|\}}$ is an orthonormal basis for $L(\mathbb{C}^2)$.

↳ Observe: there are $4 = 2^2$ elements! $\langle B_i, B_j\rangle = \delta_{i,j}$

Check: take inner products!

$$\langle |0\rangle\langle 0|, |0\rangle\langle 1|\rangle = \mathrm{Tr}\left[(|0\rangle\langle 0|)^\dagger (|0\rangle\langle 1|)\right] = \mathrm{Tr}[|0\rangle\underbrace{\langle 0|0\rangle}_{=1}\langle 1|] = \mathrm{Tr}[|0\rangle\langle 1|] = \langle 1|0\rangle = 0.$$

$$\langle |0\rangle\langle 1|, |1\rangle\langle 0|\rangle = \mathrm{Tr}\left[(|0\rangle\langle 1|)^\dagger |1\rangle\langle 0|\right] = \mathrm{Tr}[|1\rangle\underbrace{\langle 0|1\rangle}_{=0}\langle 0|] = 0.$$

$$\langle |0\rangle\langle 0|, |0\rangle\langle 0|\rangle = \mathrm{Tr}\left[(|0\rangle\langle 0|)^\dagger |0\rangle\langle 0|\right] = \mathrm{Tr}[|0\rangle\underbrace{\langle 0|0\rangle}_{=1}\langle 0|] = \mathrm{Tr}[|0\rangle\langle 0|] = \langle 0|0\rangle = 1$$

In general: $\langle |i\rangle\langle j|, |k\rangle\langle l|\rangle = \mathrm{Tr}\left[(|i\rangle\langle j|)^\dagger |k\rangle\langle l|\right] = \mathrm{Tr}[|j\rangle\underbrace{\langle i|k\rangle}_{=\delta_{i,k}}\langle l|]$

$$= \delta_{i,k}\,\mathrm{Tr}[|j\rangle\langle l|] = \delta_{i,k}\langle l|j\rangle = \delta_{i,k}\,\delta_{j,l}$$

(*) Holds for arbitrary dimension! $\{|i\rangle\langle j| : i,j \in \{0,1,\dots,d-1\}\}$ is an orthonormal basis for $L(\mathbb{C}^d)$.

- Special basis for $d=2$: Pauli Matrices (aka Pauli gates).

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\mathrm{Tr}[X] = \mathrm{Tr}[Y] = \mathrm{Tr}[Z] = 0$

$\underbrace{X^2 = \mathbb{1},\ Y^2 = \mathbb{1},\ Z^2 = \mathbb{1}}$.

↳ Bit-flip matrix: $X|0\rangle = |1\rangle$, $X|1\rangle = |0\rangle$.

$\mathrm{Tr}[X^\dagger X] = 2 = \mathrm{Tr}[Y^\dagger Y] = \mathrm{Tr}[Z^\dagger Z]$

They are Hermitian: $X^\dagger = X$, $Y^\dagger = Y$, $Z^\dagger = Z$.

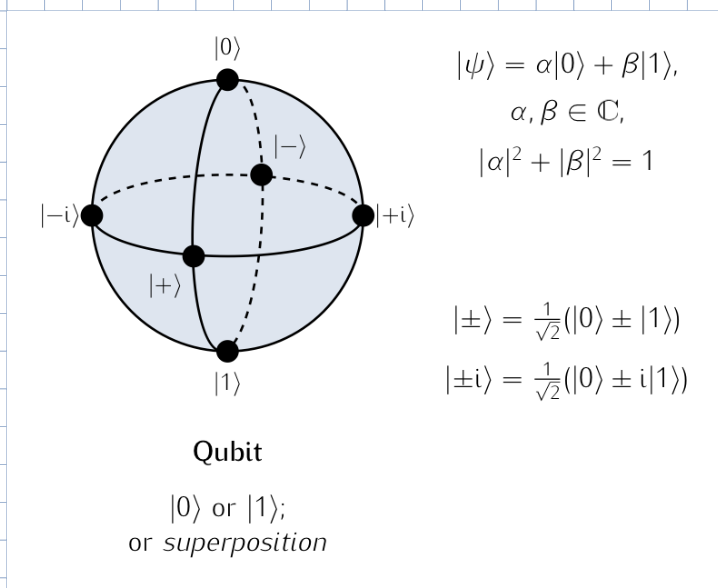
They are orthogonal: $\langle X, Y \rangle = 0$, $\langle X, Z \rangle = 0$, $\langle Z, Y \rangle = 0$.

$\{\mathbb{1}, X, Y, Z\}$ forms orthogonal basis for $L(\mathbb{C}^2)$!

$$\mathrm{Tr}[X^\dagger Y] = \mathrm{Tr}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \mathrm{Tr}\left[\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\right] = 0$$

$\Rightarrow$ Any $M \in L(\mathbb{C}^2)$ can be written as $M = \frac{1}{2}(c_0 \mathbb{1} + c_1 X + c_2 Y + c_3 Z)$.
$c_0, c_1, c_2, c_3 \in \mathbb{C}$.

3 Quantum States as Density Matrices



- We have seen state vectors:

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.

- Consider the matrix $\rho = |\psi\rangle\langle\psi|$.
It has the following properties:

(1) $\rho = \rho^\dagger$ (Hermitian).

$$|\psi\rangle\langle\psi| = (\alpha|0\rangle + \beta|1\rangle)(\bar{\alpha}\langle 0| + \bar{\beta}\langle 1|)$$
$$= |\alpha|^2 |0\rangle\langle 0| + \alpha\bar{\beta}|0\rangle\langle 1| + \beta\bar{\alpha}|1\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$$

$(M_1 + M_2)^\dagger = M_1^\dagger + M_2^\dagger$

$$(|\psi\rangle\langle\psi|)^\dagger = |\alpha|^2(|0\rangle\langle 0|)^\dagger + \underbrace{\overline{\alpha\bar{\beta}}}_{= \bar{\alpha}\cdot\bar{\bar{\beta}} = \bar{\alpha}\beta}(|0\rangle\langle 1|)^\dagger + \underbrace{\overline{\beta\bar{\alpha}}}_{= \bar{\beta}\cdot\bar{\bar{\alpha}} = \bar{\beta}\alpha}(|1\rangle\langle 0|)^\dagger + |\beta|^2(|1\rangle\langle 1|)^\dagger$$

$\left(\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2\right)$
$\bar{\bar{z}} = z$

$$= |\alpha|^2 |0\rangle\langle 0| + \bar{\alpha}\beta|1\rangle\langle 0| + \bar{\beta}\alpha|0\rangle\langle 1| + |\beta|^2 |1\rangle\langle 1|$$
$$= |\psi\rangle\langle\psi| \quad \checkmark$$

(2) $\mathrm{Tr}[\rho] = 1$ (unit trace).

$$\mathrm{Tr}\big[|\gamma\rangle\langle\gamma|\big] = \langle\gamma|\gamma\rangle = |\alpha|^2 + |\beta|^2 = 1 \quad \checkmark$$

(from above!)

(3) $\rho \geq 0$ (Positive Semi-definite).

⊛ A Hermitian matrix $M \in L(\mathbb{C}^d)$ is called positive semi-definite (denoted $M \geq 0$) if $\langle v|M|v\rangle \geq 0 \ \forall \ |v\rangle \in \mathbb{C}^d$.

This is equivalent to: all eigenvalues are non-negative

⊛ Recall eigenvalues and eigenvectors: $M|v\rangle = \lambda|v\rangle$, $M = UDU^\dagger$, $U^\dagger U = \mathbb{1}$.

diagonal matrix.

unitary

Eigenvector

Eigenvalue.

Eigenvalue (Spectral) decomposition: $M = \sum_{k=1}^{r} \lambda_k |v_k\rangle\langle v_k|$

⊛ For $\rho = |\gamma\rangle\langle\gamma| \rightarrow \langle v|\rho|v\rangle = \langle v|\gamma\rangle\langle\gamma|v\rangle = |\langle\gamma|v\rangle|^2 \geq 0 \ \forall \ |v\rangle \ \checkmark$.

- Any matrix satisfying (1), (2), (3) is called a density matrix / operator

⊛ Axiom of Quantum Mechanics: The state of any quantum system is mathematically by a density matrix. (arbitrary dimension).

- The density matrix $\rho = |\gamma\rangle\langle\gamma|$ describes a pure state. (on the surface of the Bloch sphere.)

- What about more general density matrices? Consider $d=2$.

We can write $\rho = \frac{1}{2}(c_0 \mathbb{1} + c_1 X + c_2 Y + c_3 Z)$ (b/c $\{\mathbb{1}, X, Y, Z\}$ is a basis).

We want this to satisfy: (1) $\rho^\dagger = \rho$; (2) $\mathrm{Tr}[\rho] = 1$; (3) $\rho \geq 0$.

(1) $\rho^\dagger = \frac{1}{2}(\bar{c}_0 \mathbb{1}^\dagger + \bar{c}_1 X^\dagger + \bar{c}_2 Y^\dagger + \bar{c}_3 Z^\dagger)$ $\quad (\mathbb{1}^\dagger = \mathbb{1}, X^\dagger = X, Y^\dagger = Y, Z^\dagger = Z)$

$= \frac{1}{2}(\bar{c}_0 \mathbb{1} + \bar{c}_1 X + \bar{c}_2 Y + \bar{c}_3 Z)$

$\overset{(!)}{=} \frac{1}{2}(c_0 \mathbb{1} + c_1 X + c_2 Y + c_3 Z) \iff c_0 = \bar{c}_0,\ c_1 = \bar{c}_1,\ c_2 = \bar{c}_2,\ c_3 = \bar{c}_3$ (b/c $\{\mathbb{1}, X, Y, Z\}$ is a basis).

(all are real numbers).

(2) $\rho = \frac{1}{2}(r_0 \mathbb{1} + r_1 X + r_2 Y + r_3 Z)$, $r_0, r_1, r_2, r_3 \in \mathbb{R}$.

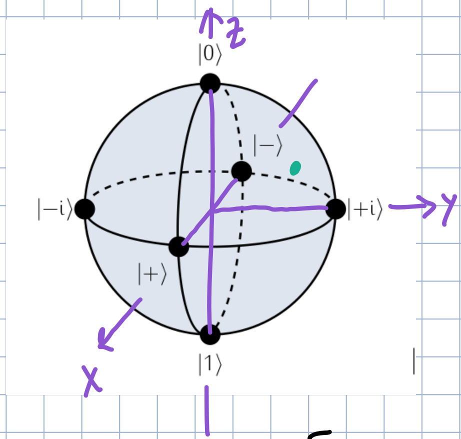
$\mathrm{Tr}[\rho] = \frac{1}{2}(r_0 \underbrace{\mathrm{Tr}[\mathbb{1}]}_{=2} + r_1 \underbrace{\mathrm{Tr}[X]}_{=0} + r_2 \underbrace{\mathrm{Tr}[Y]}_{=0} + r_3 \underbrace{\mathrm{Tr}[Z]}_{=0}) \overset{(!)}{=} 1 \iff r_0 = 1$

(3) $\rho = \frac{1}{2}(\mathbb{1} + r_1 X + r_2 Y + r_3 Z) \rightarrow$ Find the eigenvalues!

⊛ Observe:
- $\langle X, \rho \rangle = \mathrm{Tr}[X\rho] = \mathrm{Tr}[X \cdot \frac{1}{2}(\mathbb{1} + r_1 X + r_2 Y + r_3 Z)]$
$= \frac{1}{2}(\underbrace{\mathrm{Tr}[X]}_{=0} + r_1 \underbrace{\mathrm{Tr}[X^2]}_{=2} + r_2 \underbrace{\mathrm{Tr}[XY]}_{=0} + r_3 \underbrace{\mathrm{Tr}[XZ]}_{=0})$
$= r_1$
- $\langle Y, \rho \rangle = r_2$
- $\langle Z, \rho \rangle = r_3$.

$$\rho = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{pmatrix} \Rightarrow \lambda_{\pm} = \frac{1}{2}\left(1 \pm \sqrt{r_x^2+r_y^2+r_z^2}\right) \geq 0.$$

Need $\lambda_- \geq 0 \Rightarrow \frac{1}{2}\left(1 - \sqrt{r_x^2+r_y^2+r_z^2}\right) \geq 0 \Rightarrow \boxed{r_x^2 + r_y^2 + r_z^2 \leq 1}$



$\vec{r} = (r_x, r_y, r_z) \in \mathbb{R}^3$ inside the unit sphere!

Observe: $\|\rho\|_2^2 = \mathrm{Tr}[\rho^2] = \frac{1}{2}(1 + \underbrace{r_x^2 + r_y^2 + r_z^2}_{\leq 1}) \leq 1$

$\rho = \frac{1}{2}(\mathbb{1} + r_x X + r_y Y + r_z Z)$

For a pure state $\rho = |\psi\rangle\langle\psi|$, $\rho^2 = |\psi\rangle\underbrace{\langle\psi|\psi\rangle}_{=1}\langle\psi| = |\psi\rangle\langle\psi| \Rightarrow \mathrm{Tr}[\rho^2] = 1$
$\rho^2 = \rho$

$\Rightarrow r_x^2 + r_y^2 + r_z^2 = 1 \Rightarrow$ pure states are on the surface of the unit sphere!

(*) We call $\mathrm{Tr}[\rho^2]$ the **purity** of ρ $\rightarrow$ ρ pure if + only if $\mathrm{Tr}(\rho^2) = 1$.

(*) Origin, $\vec{r} = (0,0,0)$ contains the **maximally-mixed state**: $\frac{\mathbb{1}}{2}$.

$\frac{\mathbb{1}}{2} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$

(*) Points on the X-axis: $\vec{r} = (\pm 1, 0, 0) \rightarrow \rho = \frac{1}{2}(\mathbb{1} \pm X) = |\pm\rangle\langle\pm|$, $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ $= \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ \pm 1\end{pmatrix}$

These are eigenstates of X! $X|\pm\rangle = \pm|\pm\rangle$.

(*) Points on the Y-axis: $\vec{r} = (0, \pm 1, 0) \rightarrow \rho = \frac{1}{2}(\mathbb{1} \pm Y) = |\pm i\rangle\langle \pm i|$,

$|\pm i\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1 \\ \pm i\end{pmatrix}$

These are eigenstates of Y! $Y|\pm i\rangle = \pm|\pm i\rangle$.

(*) Points on the Z-axis: $\vec{r} = (0, 0, \pm 1) \rightarrow \rho = \frac{1}{2}(\mathbb{1} \pm Z)$ $\xrightarrow{(+)} |0\rangle\langle 0|$, $\xrightarrow{(-)} |1\rangle\langle 1|$. $|0\rangle = \begin{pmatrix}1 \\ 0\end{pmatrix}$, $|1\rangle = \begin{pmatrix}0 \\ 1\end{pmatrix}$.

These are eigenstates of Z! $Z|0\rangle = |0\rangle$, $Z|1\rangle = -|1\rangle$.