① Determining Entanglement.

(a) Given a state vector $|\psi\rangle_{AB}$, how to determine if it is entangled or not?
Precisely: $\overset{?}{\exists}\ |\phi_1\rangle_A, |\phi_2\rangle_B$ s.t. $|\psi\rangle_{AB} = |\phi_1\rangle_A \otimes |\phi_2\rangle_B$?

⊛ Theorem (Schmidt Decomposition): Every state vector $|\psi\rangle_{AB}$ can be written as

$$|\psi\rangle_{AB} = \sum_{k=1}^{r} s_k\, |e_k\rangle_A \otimes |f_k\rangle_B.$$

$$|\psi\rangle_{AB} = \sum_{i,j=0}^{d-1} c_{ij}\, |i\rangle_A \otimes |j\rangle_B$$

- r: Schmidt rank
- $\{s_k\}_{k=1}^{r}$: Schmidt coefficients $s_k > 0\ \forall k$
- $\{|e_k\rangle_A\}_{k=1}^{r}$ and $\{|f_k\rangle_B\}_{k=1}^{r}$ are orthonormal vectors.

(Proof (recall): Write the vector $|\psi\rangle_{AB}$ as a matrix, then do the singular value decomposition of the matrix.)

- The Schmidt rank is unique — so we get the following criterion:

> - If $r=1$, then $|\psi\rangle = |e_1\rangle \otimes |f_1\rangle \Rightarrow$ not entangled!
> - If $r>1$, then entangled! $|\Phi^+\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k\rangle_A |k\rangle_B$

(b) What about mixed states? More complicated!

- Example: convex mixtures of Bell states. $\left(|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle \pm |1,1\rangle),\ |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle \pm |1,0\rangle)\right)$

$$\rho_{AB} = p_1 \Phi^+_{AB} + p_2 \Phi^-_{AB} + p_3 \Psi^+_{AB} + p_4 \Psi^-_{AB},$$ probabilities $p_1, p_2, p_3, p_4 \in [0,1]$, $p_1 + p_2 + p_3 + p_4 = 1$

$\Phi^\pm_{AB} = |\Phi^\pm\rangle\langle\Phi^\pm|$, $\Psi^\pm_{AB} = |\Psi^\pm\rangle\langle\Psi^\pm|_{AB}$.

- $p_1 = 1,\ p_2 = p_3 = p_4 = 0 \Rightarrow \rho_{AB} = \Phi^+_{AB} \Rightarrow$ entangled!

- $p_1 = 0,\ p_2 = 1,\ p_3 = p_4 = 0 \Rightarrow \rho_{AB} = \Phi^-_{AB} \Rightarrow$ entangled!

- $p_1 = p_2 = 0,\ p_3 = 1,\ p_4 = 0 \Rightarrow \rho_{AB} = \Psi^+_{AB} \Rightarrow$ entangled!

- $p_1 = p_2 = p_3 = 0,\ p_4 = 1 \Rightarrow \rho_{AB} = \Psi^-_{AB} \Rightarrow$ entangled!

B/c Bell states form ONB.

- But: $p_1 = p_2 = p_3 = p_4 = \frac{1}{4} \Rightarrow \rho_{AB} = \frac{1}{4}\Phi^+_{AB} + \frac{1}{4}\Phi^-_{AB} + \frac{1}{4}\Psi^+_{AB} + \frac{1}{4}\Psi^-_{AB} = \frac{1}{4}\mathbb{1}_A \otimes \mathbb{1}_B$

$\Rightarrow$ not entangled!

- General criterion: Positive Partial Transpose (PPT) criterion

Recall: Transpose of a matrix (flip rows and columns).

In Bra-ket notation:

$$M = \sum_{i,j=0}^{d-1} m_{i,j} |i\rangle\langle j|$$

$$M^T = \sum_{i,j=0}^{d-1} m_{i,j} |j\rangle\langle i| \quad \hookrightarrow (M^T)^T = M.$$

For $M_{AB} \in L(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B})$:

$$M_{AB} = \sum_{i,i'=0}^{d_A-1} \sum_{j,j'=0}^{d_B-1} m_{\substack{i,j\\ i',j'}} |i\rangle\langle i'|_A \otimes |j\rangle\langle j'|_B.$$

Partial transpose on A:

$$M_{AB}^{T_A} = \sum_{i,i'=0}^{d_A-1} \sum_{j,j'=0}^{d_B-1} m_{\substack{i,j\\ i',j'}} |i'\rangle\langle i|_A \otimes |j\rangle\langle j'|_B.$$

Partial transpose on B:

$$M_{AB}^{T_B} = \sum_{i,i'=0}^{d_A-1} \sum_{j,j'=0}^{d_B-1} m_{\substack{i,j\\ i',j'}} |i\rangle\langle i'|_A \otimes |j'\rangle\langle j|_B.$$

Full transpose: $M_{AB}^T = M_{AB}^{T_A T_B} \Rightarrow M_{AB}^{T_A} = (M_{AB}^{T_B})^T$

$$\left(M_{AB}^{T_A}\right)^{T_A} = M_{AB} \qquad \left(M_{AB}^{T_B}\right)^T = \left(M_{AB}^{T_B}\right)^{T_A T_B} = M_{AB}^{T_A}$$

For $$M_{AB} = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & & & & \\ 01 & & & & \\ 10 & & & & \\ 11 & & & & \end{array}$$

$$M_{AB} = \begin{array}{cc} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ m_5 & m_6 & m_7 & m_8 \\ m_9 & m_{10} & m_{11} & m_{12} \\ m_{13} & m_{14} & m_{15} & m_{16} \end{pmatrix} \end{array}$$

$$M_{AB}^{T_B} = \begin{array}{cc} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{pmatrix} m_1 & m_5 & m_3 & m_7 \\ m_2 & m_6 & m_4 & m_8 \\ m_9 & m_{13} & m_{11} & m_{15} \\ m_{10} & m_{14} & m_{12} & m_{16} \end{pmatrix} \end{array}$$

PHYSICAL REVIEW LETTERS

VOLUME 77 | 19 AUGUST 1996 | NUMBER 8

Separability Criterion for Density Matrices

Asher Peres*
Department of Physics, Technion–Israel Institute of Technology, 32 000, Haifa, Israel
(Received 8 April 1996)

N·H ELSEVIER
25 November 1996
Physics Letters A 223 (1996) 1–8
PHYSICS LETTERS A

Separability of mixed states: necessary and sufficient conditions

Michał Horodecki [a], Paweł Horodecki [b], Ryszard Horodecki [c]
[a] *Department of Mathematics and Physics, University of Gdańsk, 80-952 Gdańsk, Poland*
[b] *Faculty of Applied Physics and Mathematics, Technical University of Gdańsk, 80-952 Gdańsk, Poland*
[c] *Institute of Theoretical Physics and Astrophysics, University of Gdańsk, 80-952 Gdańsk, Poland*

Received 24 June 1996; accepted for publication 9 September 1996
Communicated by V.M. Agranovich

Theorem (PPT Criterion): ρ_{AB} separable $\Rightarrow \rho_{AB}^{T_A} \geq 0,\ \rho_{AB}^{T_B} \geq 0$.

$X \Rightarrow Y$
$\neg Y \Rightarrow \neg X$

Contrapositive: $\rho_{AB}^{T_{A/B}} \not\geq 0 \Rightarrow \rho_{AB}$ entangled

$\Rightarrow$ Just check if $\rho_{AB}^{T_{A/B}}$ has a negative eigenvalue!

Converse: ~~$\rho_{AB}^{T_{A/B}} \geq 0 \Rightarrow \rho_{AB}$ separable~~ $\leftrightarrow$ NOT true in general!
(There are entangled states w/ positive partial transpose.)

(Example in this paper.)

4 August 1997

PHYSICS LETTERS A

Physics Letters A 232 (1997) 333–339

Separability criterion and inseparable mixed states with positive partial transposition

Paweł Horodecki

Faculty of Applied Physics and Mathematics, Technical University of Gdańsk, 80-952 Gdańsk, Poland

Received 26 March 1997; revised manuscript received 5 June 1997; accepted for publication 5 June 1997
Communicated by V.M. Agranovich

Proof: ρ_{AB} separable $\Rightarrow \rho_{AB} = \sum_x p(x)\, \tau_A^x \otimes \omega_B^x$

$$\Rightarrow \rho_{AB}^{T_B} = \sum_x p(x)\, \underbrace{\tau_A^x}_{\geq 0} \otimes \underbrace{(\omega_B^x)^T}_{\geq 0} \geq 0$$

$M \geq 0 \Rightarrow M^T \geq 0$ (full transpose does not change eigenvalues.).

$\Rightarrow \rho_{AB}^{T_B} \geq 0$. $\rho_{AB}^{T_A} = (\rho_{AB}^{T_B})^T$ ← Full transpose. $\Rightarrow \rho_{AB}^{T_A} \geq 0$. ∎

⊛ Converse is true for $2\otimes 2$ and $2\otimes 3$!

↳ ρ_{AB} separable $\Leftrightarrow \rho_{AB}^{T_{A/B}} \geq 0$.

- Examples of the PPT criterion: $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$.

$$- \rho_{AB} = \Phi^+_{AB} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \rho_{AB}^{T_B} = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

$\mathbb{F}_{AB}$ → swap operator.

⊛ Eigenvalues of $\mathbb{F}_{AB}$ are ± 1.

⊛ $\mathbb{F}_{AB} = \sum_{i,j=0}^{d-1} |j,i\rangle\langle i,j|$ $\quad (\mathbb{F}_{AB}|i,j\rangle = |j,i\rangle)$

$|j,i\rangle\langle i,j| \equiv |j\rangle\langle i| \otimes |i\rangle\langle j|$

$\mathbb{F}_{AB}|0\rangle|0\rangle = |0\rangle|0\rangle$
$\mathbb{F}_{AB}|0\rangle|1\rangle = |1\rangle|0\rangle$
$\mathbb{F}_{AB}|1\rangle|0\rangle = |0\rangle|1\rangle$
$\mathbb{F}_{AB}|1\rangle|1\rangle = |1\rangle|1\rangle$.

(*) $\mathbb{F}_{AB}\left(|\psi_1\rangle_A \otimes |\psi_2\rangle_B\right) = |\psi_2\rangle_A \otimes |\psi_1\rangle_B$.

Proof: $\mathbb{F}_{AB}\left(|\psi_1\rangle_A \otimes |\psi_2\rangle_B\right) = \left(\sum_{i,j=0}^{d-1} |j\rangle\langle i|_A \otimes |i\rangle\langle j|_B\right)\left(|\psi_1\rangle_A \otimes |\psi_2\rangle_B\right)$

$$= \sum_{i,j=0}^{d-1} |j\rangle\langle i|\psi_1\rangle \otimes |i\rangle\langle j|\psi_2\rangle$$

$$= \sum_{i,j=0}^{d-1} |j\rangle\langle j|\psi_2\rangle \otimes |i\rangle\langle i|\psi_1\rangle$$

$$= \underbrace{\left(\sum_{j=0}^{d-1} |j\rangle\langle j|_A\right)}_{=\mathbb{1}_A}|\psi_2\rangle_A \otimes \underbrace{\left(\sum_{i=0}^{d-1} |i\rangle\langle i|_B\right)}_{=\mathbb{1}_B}|\psi_1\rangle_B$$

$$= |\psi_2\rangle_A \otimes |\psi_1\rangle_B.$$

$p_1 = 1-p$

$p_2 = p_3 = p_4 = \frac{p}{3}$.

- Isotropic state: $\rho_{AB}^{(p)} = (1-p)\,\Phi^+_{AB} + \frac{p}{3}\left(\Phi^-_{AB} + \Psi^+_{AB} + \Psi^-_{AB}\right)$, $p \in [0,1]$.

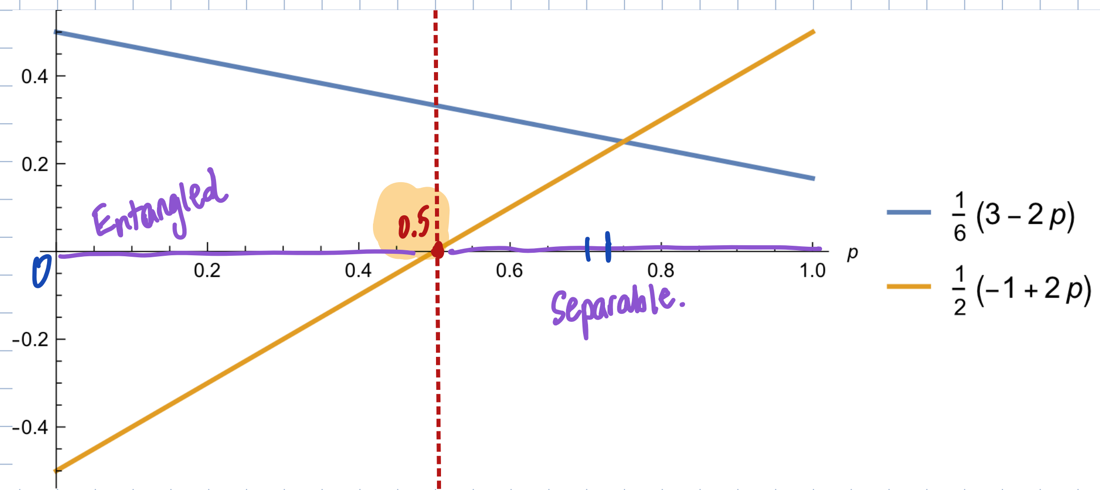
$$\rho_{AB}^{(p)} = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & \frac{1-p}{2}+\frac{p}{6} & 0 & 0 & \frac{1-p}{2}-\frac{p}{6} \\ 01 & 0 & \frac{p}{3} & 0 & 0 \\ 10 & 0 & 0 & \frac{p}{3} & 0 \\ 11 & \frac{1-p}{2}-\frac{p}{6} & 0 & 0 & \frac{1-p}{2}+\frac{p}{6} \end{array}$$

$$\left(\rho_{AB}^{(p)}\right)^{T_B} = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & \frac{1-p}{2}+\frac{p}{6} & 0 & 0 & 0 \\ 01 & 0 & \frac{p}{3} & \frac{1-p}{2}-\frac{p}{6} & 0 \\ 10 & 0 & \frac{1-p}{2}-\frac{p}{6} & \frac{p}{3} & 0 \\ 11 & 0 & 0 & 0 & \frac{1-p}{2}+\frac{p}{6} \end{array}$$

Eigenvalues of $\left(\rho_{AB}^{(p)}\right)^{T_B}$: $\lambda_1 = \frac{1}{6}(3-2p)$ (multiplicity 3)

$\lambda_2 = \frac{1}{2}(-1+2p)$ (multiplicity 1).

(* We need a negative eigenvalue for entanglement!)



$$\rho_{AB}^{(p)} = (1-p)\Phi_{AB}^{+} + \frac{p}{3}\left(\Phi_{AB}^{-} + \Psi_{AB}^{+} + \Psi_{AB}^{-}\right)$$

(*) Observe: $\mathrm{Tr}\left[\rho_{AB}^{(p)}\Phi_{AB}^{+}\right] = \langle\Phi^{+}|\rho_{AB}^{(p)}|\Phi^{+}\rangle$

$$= (1-p)\langle\Phi^{+}|\Phi_{AB}^{+}|\Phi^{+}\rangle + \frac{p}{3}\left(\cancelto{0}{\langle\Phi^{+}|\Phi_{AB}^{-}|\Phi^{+}\rangle} + \cancelto{0}{\langle\Phi^{+}|\Psi_{AB}^{+}|\Phi^{+}\rangle} + \cancelto{0}{\langle\Phi^{+}|\Psi_{AB}^{-}|\Phi^{+}\rangle}\right)$$

$$= 1-p \Rightarrow p = 1 - \langle\Phi^{+}|\rho_{AB}^{(p)}|\Phi^{+}\rangle.$$

↳ This is called the fidelity.

(*) So $\rho_{AB}^{(p)}$ is entangled if and only if $\langle\Phi^{+}|\rho_{AB}^{(p)}|\Phi^{+}\rangle > \frac{1}{2}$

① Classical vs. Quantum Correlations: Bell/CHSH Inequality

- The above criteria for detecting entanglement are purely mathematical.
- In practice, we need to do measurements to determine whether or not qubits are entangled.

- <u>Recall example from before:</u> $\rho_{AB} = \frac{1}{2}\left(|0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_B + |1\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B\right)$
  - This state is separable (not entangled).
  - If Alice and Bob both measure in $\{|0\rangle, |1\rangle\}$ basis, then:
    - They both get "0" and "1" w/ probability $\frac{1}{2}$.
      (i.e., locally, it looks completely random, b/c $\rho_A = \mathrm{Tr}_B[\rho_{AB}] = \frac{1}{2}\mathbb{1}_A$ and $\rho_B = \mathrm{Tr}_A(\rho_{AB}) = \frac{1}{2}\mathbb{1}_B$.)
    - But if they compare their measurement outcomes, they will always be the opposite! (Whenever Alice got a "0" or "1", Bob got "1" or "0".) So their outcomes are perfectly <u>anti-correlated</u>.

- But the same thing happens with the maximally-entangled state!

  $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(\underbrace{|0\rangle_A}_{|e_1\rangle}\underbrace{|1\rangle_B}_{|f_1\rangle} - \underbrace{|1\rangle_A}_{|e_2\rangle}\underbrace{|0\rangle_B}_{|f_2\rangle})$ — in the $\{|0\rangle, |1\rangle\}$ basis, the outcomes are perfectly anti-correlated!

- So how to distinguish the two? What makes the entangled state special?

- To see the difference, measure in a different basis! Say in the $\{|+\rangle, |-\rangle\}$ basis.
  - For $\rho_{AB} = \frac{1}{2}\left(|0\rangle\langle 0|_A \otimes |1\rangle\langle 1|_B + |1\rangle\langle 1|_A \otimes |0\rangle\langle 0|_B\right)$:

    $$\Pr[+,+] = \mathrm{Tr}[(|+\rangle\langle +|_A \otimes |+\rangle\langle +|_B)\rho_{AB}] = \frac{1}{2}\left(\underbrace{\langle +|0\rangle}_{\frac{1}{\sqrt{2}}}\langle 0|+\rangle\langle +|1\rangle\langle 1|+\rangle + \underbrace{\langle +|1\rangle}_{\frac{1}{\sqrt{2}}}\langle 1|+\rangle\langle +|0\rangle\langle 0|+\rangle\right)$$

    $$\left(|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)\right) \qquad = \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\right) = \frac{1}{4}.$$

    Similarly: $\Pr[+,-] = \Pr[-,+] = \Pr[-,-] = \frac{1}{4}$.

    So the outcomes are completely <u>uncorrelated</u>!

    Why? B/c the joint distribution is a product of the marginal distributions.

- Marginal distributions are given by the partial trace:
  - Marginal for Alice: $\rho_A = \frac{1}{2}\mathbb{1}_A \Rightarrow Pr_A[+] = Pr_A[-] = \frac{1}{2}$
  - Marginal for Bob: $\rho_B = \frac{1}{2}\mathbb{1}_B \Rightarrow Pr_B[+] = Pr_B[-] = \frac{1}{2}$
  - Product of the marginals: $Pr_A[+]\cdot Pr_B[+] = \frac{1}{4}$, $Pr_A[+]\cdot Pr_B[-] = \frac{1}{4}$

$P_{XY} = P_X \cdot P_Y$ $\qquad Pr_A[-]\cdot Pr_B[+] = \frac{1}{4}$, $Pr_A[-]\cdot Pr_B[-] = \frac{1}{4}$.

This is the same as the joint distribution calculated above!
So Alice and Bob's random variables are independent $\Rightarrow$ no correlation.

- But for $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B)$, the outcomes are still perfectly anti-correlated!
  - $(\langle +|_A \otimes \langle +|_B)|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(\underbrace{\langle +|0\rangle}_{=\frac{1}{\sqrt{2}}}\underbrace{\langle +|1\rangle}_{\frac{1}{\sqrt{2}}} - \langle +|1\rangle\langle +|0\rangle) = 0 \Rightarrow \underline{Pr[+,+] = 0}$
  - $(\langle +|_A \otimes \langle -|_B)|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(\underbrace{\langle +|0\rangle}_{\frac{1}{\sqrt{2}}}\underbrace{\langle -|1\rangle}_{-\frac{1}{\sqrt{2}}} - \underbrace{\langle +|1\rangle}_{\frac{1}{\sqrt{2}}}\underbrace{\langle -|0\rangle}_{\frac{1}{\sqrt{2}}}) = \frac{1}{\sqrt{2}}(-\frac{1}{2}-\frac{1}{2}) = -\frac{1}{\sqrt{2}}$

    $\Rightarrow \underline{Pr[+,-] = |(\langle +|_A \otimes \langle -|_B)|\Psi^-\rangle_{AB}|^2} = \frac{1}{2}$
  - $(\langle -|_A \otimes \langle +|_B)|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(\underbrace{\langle -|0\rangle}_{\frac{1}{\sqrt{2}}}\underbrace{\langle +|1\rangle}_{\frac{1}{\sqrt{2}}} - \underbrace{\langle -|1\rangle}_{-\frac{1}{\sqrt{2}}}\underbrace{\langle +|0\rangle}_{\frac{1}{\sqrt{2}}}) = \frac{1}{\sqrt{2}}(\frac{1}{2}+\frac{1}{2}) = \frac{1}{\sqrt{2}}$

    $\Rightarrow \underline{Pr[-,+] = |(\langle -|_A \otimes \langle +|_B)|\Psi^-\rangle_{AB}|^2} = \frac{1}{2}$
  - $(\langle -|_A \otimes \langle -|_B)|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(\underbrace{\langle -|0\rangle}_{=\frac{1}{\sqrt{2}}}\underbrace{\langle -|1\rangle}_{-\frac{1}{\sqrt{2}}} - \underbrace{\langle -|1\rangle}_{-\frac{1}{\sqrt{2}}}\underbrace{\langle -|0\rangle}_{\frac{1}{\sqrt{2}}}) = 0 \Rightarrow \underline{Pr[-,-] = 0}.$

- The anti-correlation exists for <u>any</u> basis measurement!

  - <u>Fact</u>: Let U be an arbitrary 2×2 unitary matrix. Then

$$(U\otimes U)|\Psi^-\rangle\langle\Psi^-|(U\otimes U)^\dagger = |\Psi^-\rangle\langle\Psi^-|.$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle)$$

<u>Proof</u>: First consider an arbitrary 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a,b,c,d \in \mathbb{C}$.

$M|0\rangle = a|0\rangle + c|1\rangle$

$M|1\rangle = b|0\rangle + d|1\rangle$

$$
\begin{aligned}
(M\otimes M)|\Psi^-\rangle &= \frac{1}{\sqrt{2}}\left(M|0\rangle\otimes M|1\rangle - M|1\rangle\otimes M|0\rangle\right)\\
&= \frac{1}{\sqrt{2}}\Big((a|0\rangle + c|1\rangle)\otimes(b|0\rangle + d|1\rangle) - (b|0\rangle + d|1\rangle)\otimes(a|0\rangle + c|1\rangle)\Big)\\
&= \frac{1}{\sqrt{2}}\Big(\cancel{ab|0,0\rangle} + ad|0,1\rangle + cb|1,0\rangle + \cancel{cd|1,1\rangle}\\
&\qquad - \cancel{ab|0,0\rangle} - cb|1,0\rangle - ad|1,0\rangle - \cancel{cd|1,1\rangle}\Big)\\
&= \frac{1}{\sqrt{2}}\Big((ad-bc)|0,1\rangle - (ad-bc)|1,0\rangle\Big)\\
&= \frac{1}{\sqrt{2}}(ad-bc)\left(|0,1\rangle - |1,0\rangle\right)\\
&= (ad-bc)\frac{1}{\sqrt{2}}\left(|0,1\rangle - |1,0\rangle\right).\\
&= \det(M)\,|\Psi^-\rangle. \quad \rightarrow \text{ determinant of } M!
\end{aligned}
$$

So for any matrix M: $(M\otimes M)|\Psi^-\rangle\langle\Psi^-|(M\otimes M)^\dagger = |\det(M)|^2 |\Psi^-\rangle\langle\Psi^-|$.

Determinant is product of the eigenvalues.

Let $U = \sum_{k=1}^{d} \lambda_k |\psi_k\rangle\langle\psi_k|$ be the spectral decomposition of a unitary.

Then $U^\dagger = \sum_{k=1}^{d} \bar{\lambda}_k |\psi_k\rangle\langle\psi_k| \Rightarrow UU^\dagger = \sum_{k=1}^{d} |\lambda_k|^2 |\psi_k\rangle\langle\psi_k| = \mathbb{1}$ (by unitarity) $\lambda_k = re^{i\theta}$

Also, $\sum_{k=1}^{d} |\psi_k\rangle\langle\psi_k| = \mathbb{1} \Rightarrow |\lambda_k|^2 = 1 \;\forall k \Rightarrow \lambda_k = e^{i\theta_k} \Rightarrow$ eigenvalues of a unitary are complex numbers with unit modulus.

$\Rightarrow \det(U) = \lambda_1 \lambda_2 \cdots \lambda_d$ is a complex number with unit modulus.

$\Rightarrow |\det(U)|^2 = 1.$

$\Rightarrow (U \otimes U) |\Psi^-\rangle\langle\Psi^-| (U \otimes U)^\dagger = |\det(U)|^2 |\Psi^-\rangle\langle\Psi^-| = |\Psi^-\rangle\langle\Psi^-|.$ ∎

- The vectors $\{U^\dagger|0\rangle, U^\dagger|1\rangle\}$ define a measurement, with measurement operators $M_0 = U^\dagger|0\rangle\langle 0|U$, $M_1 = U^\dagger|1\rangle\langle 1|U$.

$$M_0 + M_1 = U^\dagger|0\rangle\langle 0|U + U^\dagger|1\rangle\langle 1|U = U^\dagger\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{=\mathbb{1}}U = U^\dagger U = \mathbb{1}. \checkmark$$

Probability distribution is:

$$\Pr(i,j) = \mathrm{Tr}[(M_i \otimes M_j)|\Psi^-\rangle\langle\Psi^-|_{AB}] = \mathrm{Tr}[(U^\dagger|i\rangle\langle i|U \otimes U^\dagger|j\rangle\langle j|U)|\Psi^-\rangle\langle\Psi^-|_{AB})$$

$(X_1 \otimes Y_1)(X_2 \otimes Y_2) = X_1X_2 \otimes Y_1Y_2$

$$= \mathrm{Tr}[(U^\dagger \otimes U^\dagger)(|i,j\rangle\langle i,j|)(U \otimes U)|\Psi^-\rangle\langle\Psi^-|_{AB}]$$

$$= \mathrm{Tr}[|i,j\rangle\langle i,j| \underbrace{(U \otimes U)|\Psi^-\rangle\langle\Psi^-|(U \otimes U)^\dagger}_{=|\Psi^-\rangle\langle\Psi^-| \text{ (from above)}}]$$

$$= \mathrm{Tr}[|i,j\rangle\langle i,j|\Psi^-\rangle\langle\Psi^-|]$$

$\Rightarrow$ Same distribution as the $\{|0\rangle, |1\rangle\}$ basis! So still anti-correlated!

(*) We can formalize this idea into an experimental test for entanglement.