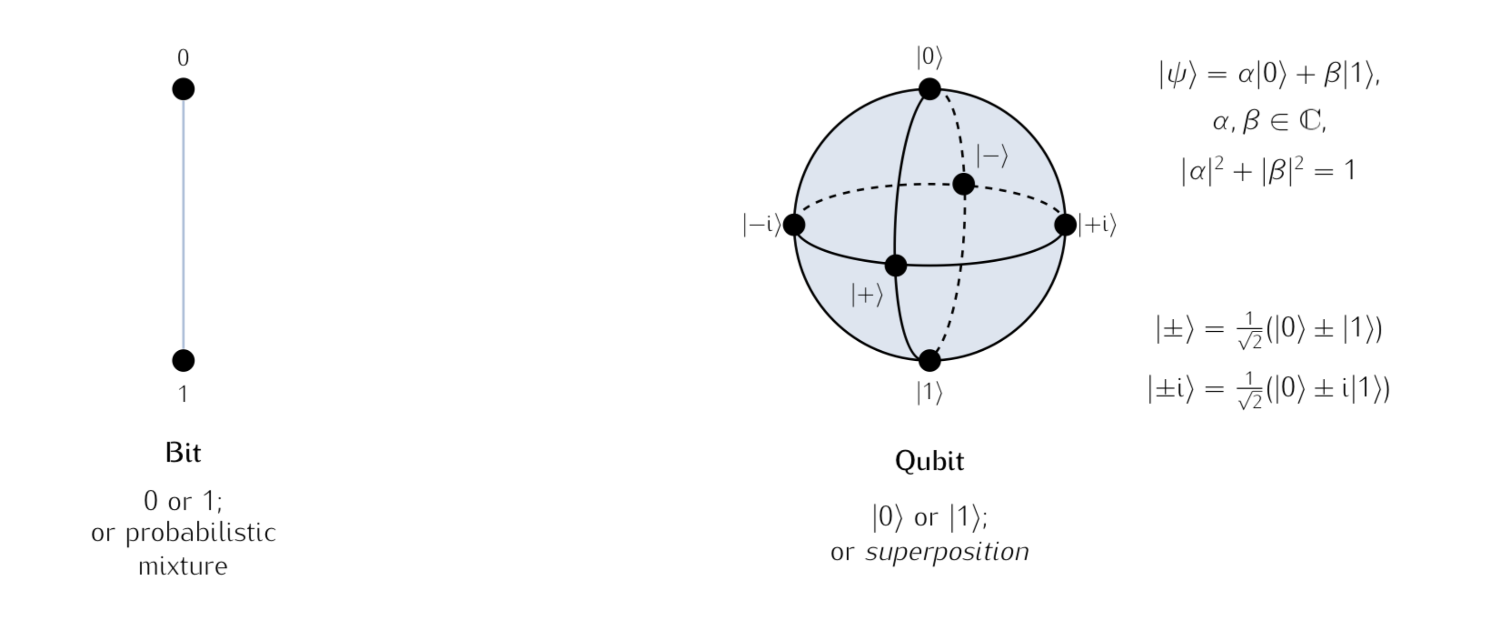
① Recap



⊛ States of live on the surface of the Bloch sphere. — they are described by 2-dimensional state vectors.

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}, \quad \underbrace{|\alpha|^2}_{Pr[0]} + \underbrace{|\beta|^2}_{Pr[1]} = 1.$$

$$\to |\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(States inside the sphere are also possible — they are described by density matrices. We will see this later...)

② Complex Vector Spaces

Complex numbers!

$$|v\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \to |v\rangle = \sum_{k=1}^{d} a_k |e_k\rangle, \quad |e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots, \quad |e_d\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\langle v| = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_d \end{pmatrix} \to \langle v| = \sum_{k=1}^{d} \bar{a}_k \langle e_k|.$$

$$|v\rangle = \sum_{k=0}^{d-1} a_k |k\rangle, \quad |u\rangle = \sum_{k=0}^{d-1} b_k |k\rangle$$

$$\langle v|u\rangle = \left(\sum_{k=0}^{d-1} \bar{a}_k \langle k|\right)\left(\sum_{k'=0}^{d-1} b_{k'} |k'\rangle\right)$$

$$|v\rangle = \sum_{k=0}^{d-1} a_k |k\rangle \rightarrow \langle k'|v\rangle = \langle k'| \left(\sum_{k=0}^{d-1} a_k |k\rangle \right)$$

$$= \sum_{k=0}^{d-1} a_k \underbrace{\langle k'|k\rangle}_{\delta_{k,k'}}$$

$$= a_{k'}$$

$$= \sum_{k,k'=0}^{d-1} \overline{a_k}\, b_{k'} \underbrace{\langle k|k'\rangle}_{\delta_{k,k'}}$$

$$= \sum_{k=0}^{d-1} \overline{a_k}\, b_k .$$

⊛ Notation: $|e_1\rangle \equiv |0\rangle, |e_2\rangle \equiv |1\rangle, \dots, |e_d\rangle \equiv |d-1\rangle$ → The "standard basis" or "computational basis".

For a qubit: $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha|0\rangle + \beta|1\rangle.$

$\alpha = \langle 0|\psi\rangle, \ \beta = \langle 1|\psi\rangle$

- Addition of vectors:

$$|v_1\rangle = \sum_{k=0}^{d-1} a_k |k\rangle, \quad |v_2\rangle = \sum_{k=0}^{d-1} b_k |k\rangle \Rightarrow |v_1\rangle + |v_2\rangle = \sum_{k=0}^{d-1} (a_k + b_k)|k\rangle.$$

- Inner product of vectors is given by $\langle v_1|v_2\rangle$.

$$|v_1\rangle = \sum_{k=0}^{d-1} a_k |k\rangle, \quad |v_2\rangle = \sum_{k=0}^{d-1} b_k |k\rangle \rightarrow \langle v_1|v_2\rangle = \underbrace{\left(\sum_{k=0}^{d-1} \bar{a}_k \langle k|\right)}_{\langle v_1|} \underbrace{\left(\sum_{k'=0}^{d-1} b_{k'}|k'\rangle\right)}_{|v_2\rangle}$$

$$\delta_{k,k'} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{if } k \neq k' \end{cases}$$

$$= \sum_{k,k'=0}^{d-1} \bar{a}_k b_{k'} \underbrace{\langle k|k'\rangle}_{=\delta_{k,k'}} = \sum_{k=0}^{d-1} \bar{a}_k b_k.$$

- The norm of a vector is $\| |v\rangle \| = \sqrt{\langle v|v\rangle} = \left(\sum_{k=0}^{d-1} \underbrace{|a_k|^2}_{a_k \cdot \bar{a}_k}\right)^{1/2}.$

$$\left(|v\rangle = \sum_{k=0}^{d-1} a_k |k\rangle\right)$$

We call a vector "normalized" if $\| |v\rangle \| = 1.$

⊛ To normalize a vector, just divide by its norm!

$$|v\rangle = \sum_{k=0}^{d-1} a_k |k\rangle \rightarrow \| |v\rangle \| = \left(\sum_{k=0}^{d-1} |a_k|^2\right)^{1/2} \Rightarrow |\tilde{v}\rangle = \frac{1}{\| |v\rangle \|} \cdot |v\rangle = \frac{1}{\| |v\rangle \|} \sum_{k=0}^{d-1} a_k |k\rangle.$$

$$= \sum_{k=0}^{d-1} \frac{a_k}{\| |v\rangle \|} |k\rangle$$

Check: $\langle \tilde{v}|\tilde{v}\rangle = \left(\frac{1}{\| |v\rangle \|} \sum_{k=0}^{d-1} \bar{a}_k \langle k|\right) \left(\frac{1}{\| |v\rangle \|} \sum_{k'=0}^{d-1} a_{k'} |k'\rangle\right)$

$$= \frac{1}{\| |v\rangle \|^2} \underbrace{\sum_{k=0}^{d-1} |a_k|^2}_{= \| |v\rangle \|^2}$$

$$= \frac{1}{\| |v\rangle \|^2} \cdot \| |v\rangle \|^2$$

$$= 1 \checkmark$$

- <u>Tensor product of vectors</u>: Used to describe the state of multiple qubits. (and entanglement!).

  - One qubit: 2-dimensional vector: $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$.

  - If qubit 1 has state $|\psi_1\rangle$ and qubit 2 has state $|\psi_2\rangle$, then the <u>joint state</u> is given by

$$|\psi_1\rangle \otimes |\psi_2\rangle = \underbrace{\begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}}_{|\psi_1\rangle} \otimes \underbrace{\begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}}_{|\psi_2\rangle} = \begin{pmatrix} \alpha_1 \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \\ \beta_1 \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \alpha_1 \alpha_2 \\ \alpha_1 \beta_2 \\ \beta_1 \alpha_2 \\ \beta_1 \beta_2 \end{pmatrix} \end{array} \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array}$$

(4-dimensional vector!)

tensor / Kronecker product

(*) Abstract notation: $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$, $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$$\Rightarrow |\psi_1\rangle \otimes |\psi_2\rangle = (\alpha_1|0\rangle + \beta_1|1\rangle) \otimes (\alpha_2|0\rangle + \beta_2|1\rangle)$$

$$= \alpha_1\alpha_2 |0\rangle \otimes |0\rangle + \alpha_1\beta_2 |0\rangle \otimes |1\rangle + \beta_1\alpha_2 |1\rangle \otimes |0\rangle + \beta_1\beta_2 |1\rangle \otimes |1\rangle$$

<u>Recall</u>: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\Rightarrow$ $\underline{|0\rangle \otimes |0\rangle} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array}$; $\underline{|0\rangle \otimes |1\rangle} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$|1\rangle \otimes |0\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $|1\rangle \otimes |1\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\rightarrow$ This defines the vector space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

⊛ $\{|0\rangle \otimes |0\rangle, |0\rangle \otimes |1\rangle, |1\rangle \otimes |0\rangle, |1\rangle \otimes |1\rangle\}$ is the standard/computational basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$. (Just all two-bit strings!)

⊛ Probabilities: $\Pr[0,0] = |\alpha_1 \alpha_2|^2$, $\Pr[0,1] = |\alpha_1 \beta_2|^2$, $\Pr[1,0] = |\beta_1 \alpha_2|^2$, $\Pr[1,1] = |\beta_1 \beta_2|^2$.

⊛ This extends to any dimension!

For $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, basis is $\{|k_1\rangle \otimes |k_2\rangle : k_1 \in \{0,1,\dots,d_1-1\}, k_2 \in \{0,1,\dots,d_2-1\}\}$

Dimension is $d_1 \cdot d_2$.

⊛ Shorthand notation: $|k_1\rangle \otimes |k_2\rangle \equiv |k_1\rangle|k_2\rangle \equiv |k_1, k_2\rangle$.

- Tensor product of three vectors:

$$|\gamma_1\rangle = \begin{pmatrix} \alpha_1 \\ \beta_1 \end{pmatrix}, \quad |\gamma_2\rangle = \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix}, \quad |\gamma_3\rangle = \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix}$$

$$\Rightarrow |\gamma_1\rangle \otimes |\gamma_2\rangle \otimes |\gamma_3\rangle = \underbrace{\left(|\gamma_1\rangle \otimes |\gamma_2\rangle\right)}\otimes |\gamma_3\rangle = |\gamma_1\rangle \otimes \left(|\gamma_2\rangle \otimes |\gamma_3\rangle\right).$$

$$= \begin{pmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \end{pmatrix} \otimes \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} = \begin{pmatrix} \alpha_1\alpha_2 \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} \\ \alpha_1\beta_2 \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} \\ \beta_1\alpha_2 \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} \\ \beta_1\beta_2 \begin{pmatrix} \alpha_3 \\ \beta_3 \end{pmatrix} \end{pmatrix} = \begin{array}{c} \begin{pmatrix} \alpha_1\alpha_2\alpha_3 \\ \alpha_1\alpha_2\beta_3 \\ \alpha_1\beta_2\alpha_3 \\ \alpha_1\beta_2\beta_3 \\ \beta_1\alpha_2\alpha_3 \\ \beta_1\alpha_2\beta_3 \\ \beta_1\beta_2\alpha_3 \\ \beta_1\beta_2\beta_3 \end{pmatrix} \end{array} \begin{array}{c} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array}$$

⊛ The resulting vector has dimension $8 = 2 \cdot 2 \cdot 2 = 2^3$!

⊛ The vector space is $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$.

⊛ Abstract notation: $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$, $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$, $|\psi_3\rangle = \alpha_3|0\rangle + \beta_3|1\rangle$

$$\Rightarrow |\psi_1\rangle \otimes |\psi_2\rangle \otimes |\psi_3\rangle = \left(\alpha_1|0\rangle + \beta_1|1\rangle\right) \otimes \left(\alpha_2|0\rangle + \beta_2|1\rangle\right) \otimes \left(\alpha_3|0\rangle + \beta_3|1\rangle\right).$$

$$= \left(\alpha_1\alpha_2|0,0\rangle + \alpha_1\beta_2|0,1\rangle + \beta_1\alpha_2|1,0\rangle + \beta_1\beta_2|1,1\rangle\right) \otimes \left(\alpha_3|0\rangle + \beta_3|1\rangle\right)$$

$$= \alpha_1\alpha_2\alpha_3|0,0,0\rangle + \alpha_1\alpha_2\beta_3|0,0,1\rangle + \alpha_1\beta_2\alpha_3|0,1,0\rangle + \alpha_1\beta_2\beta_3|0,1,1\rangle$$
$$+ \beta_1\alpha_2\alpha_3|1,0,0\rangle + \beta_1\alpha_2\beta_3|1,0,1\rangle + \beta_1\beta_2\alpha_3|1,1,0\rangle + \beta_1\beta_2\beta_3|1,1,1\rangle$$

- Arbitrary number of tensor products:

$$|\psi_j\rangle = \sum_{k=0}^{1} c_{j,k}|k\rangle = c_{j,0}|0\rangle + c_{j,1}|1\rangle$$

$$\rightarrow \underbrace{|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle}_{\text{vector in } (\mathbb{C}^2)^{\otimes n}.} = \bigotimes_{j=1}^{n} |\psi_j\rangle = \sum_{k_1,k_2,\ldots,k_n=0}^{1} c_{1,k_1} c_{2,k_2} \cdots c_{n,k_n} |k_1,k_2,\ldots,k_n\rangle$$

⊛ Generalize to arbitrary dimension! $|\psi_j\rangle = \sum_{k=0}^{d_j-1} c_{j,k}|k\rangle$

$$\Rightarrow \underbrace{\bigotimes_{j=1}^{n} |\psi_j\rangle}_{\text{Vector in } \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}.} = \sum_{k_1=0}^{d_1-1} \sum_{k_2=0}^{d_2-1} \cdots \sum_{k_n=0}^{d_n-1} c_{1,k_1} c_{2,k_2} \cdots c_{n,k_n} |k_1,k_2,\ldots,k_n\rangle$$

- Basis of n-qubit space $(\mathbb{C}^2)^{\otimes n}$ is labeled by all n-bit strings.

$$(\mathbb{C}^2)^{\otimes n} = \text{span}\{|\vec{x}\rangle : \vec{x} \in \underbrace{\{0,1\}^n}_{\text{Set of all n-bit strings.}}\}$$

Notation: $\vec{x} \in \{0,1\}^n \Rightarrow \vec{x} = (x_1, x_2, \ldots, x_n)$, $x_i \in \{0,1\}$

$|\vec{x}\rangle = |x_1, x_2, \ldots, x_n\rangle \equiv |x_1\rangle \otimes |x_2\rangle \otimes \cdots \otimes |x_n\rangle$

• Example: $n=2 \rightarrow \{ |0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle \}$

$n=3 \rightarrow \{ |0,0,0\rangle, |0,0,1\rangle, |0,1,0\rangle, |0,1,1\rangle, |1,0,0\rangle, |1,0,1\rangle, |1,1,0\rangle, |1,1,1\rangle \}$.

– Tensor product and inner product.

By definition: $\left(\langle v_1| \otimes \langle v_2|\right)\left(|u_1\rangle \otimes |u_2\rangle\right) = \langle v_1|u_1\rangle \cdot \langle v_2|u_2\rangle$.

$\langle 0,1,0|1,1,0\rangle = \underbrace{\langle 0|1\rangle}_{=0} \underbrace{\langle 1|1\rangle}_{=1} \underbrace{\langle 0|0\rangle}_{=1}$

• Example: for basis vectors, $\langle 0,1,0|1,0,1\rangle = \underbrace{\langle 0|1\rangle}_{=0} \cdot \underbrace{\langle 1|0\rangle}_{=0} \cdot \underbrace{\langle 0|1\rangle}_{=0} = 0$.

In general: for basis vectors $|\vec{x}\rangle, |\vec{y}\rangle$, $\vec{x} = (x_1, x_2, \dots, x_n) \in \{0,1\}^n$
$\vec{y} = (y_1, y_2, \dots, y_n) \in \{0,1\}^n$

$$\langle \vec{x}|\vec{y}\rangle = \delta_{\vec{x},\vec{y}} = \delta_{x_1,y_1}\delta_{x_2,y_2}\cdots\delta_{x_n,y_n}$$

• Probabilities: For a state of n qubits given by

$$|\psi\rangle = \sum_{\vec{x}\in\{0,1\}^n} c_{\vec{x}} |\vec{x}\rangle \rightarrow \text{Normalization condition is } \sum_{\vec{x}\in\{0,1\}^n} |c_{\vec{x}}|^2 = 1.$$

Probabilities are: $\Pr[\vec{x}] = |c_{\vec{x}}|^2 = \langle \vec{x}|\psi\rangle$

③ Matrices: Linear Transformations of Vectors

⊛ Matrices will be used to define and describe states inside the Bloch sphere (i.e., mixed states) and the gates we can apply to qubits on a quantum computer.

⊛ We also call matrices <u>linear operators / transformations</u>.

- <u>2x2 matrices:</u> $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $a, b, c, d \in \mathbb{C}$.

  - Multiplying a matrix with a vector: $|v\rangle = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$M|v\rangle = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix}$$

  - Matrices act linearly as follows: $M(\alpha|v_1\rangle + \beta|v_2\rangle) = \alpha M|v_1\rangle + \beta M|v_2\rangle$

$$\alpha M = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

  - Matrix multiplication: $M_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$, $M_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$

$$M_1 \cdot M_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}.$$

  - Abstract notation from vectors extends to matrices!

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

⊛ <u>Recall</u>: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\rightarrow$ $\langle 0| = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\langle 1| = \begin{pmatrix} 0 & 1 \end{pmatrix}$

⊛ <u>Observe</u>: $|0\rangle\langle 0| = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{2\times 1}\underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{1\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $|0\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$$|1\rangle\langle 0| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, \quad |1\rangle\langle 1| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

Then $\underline{M = a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|}$ $\quad a = \langle 0|M|0\rangle,\ c = \langle 1|M|0\rangle$
$\quad b = \langle 0|M|1\rangle \quad d = \langle 1|M|1\rangle$

This makes multiplication easier! (No need to remember matrix multiplication rules).

$$M|v\rangle = \left(a|0\rangle\langle 0| + b|0\rangle\langle 1| + c|1\rangle\langle 0| + d|1\rangle\langle 1|\right)\left(v_1|0\rangle + v_2|1\rangle\right)$$

$$= av_1|0\rangle\cancelto{1}{\langle 0|0\rangle} + av_2|0\rangle\cancelto{0}{\langle 0|1\rangle} + bv_1|0\rangle\cancelto{0}{\langle 1|0\rangle} + bv_2|0\rangle\cancelto{1}{\langle 1|1\rangle}$$

$$+ cv_1|1\rangle\cancelto{1}{\langle 0|0\rangle} + cv_2|1\rangle\cancelto{0}{\langle 0|1\rangle} + dv_1|1\rangle\cancelto{0}{\langle 1|0\rangle} + dv_2|1\rangle\cancelto{1}{\langle 1|1\rangle}$$

$$= \underline{(av_1 + bv_2)|0\rangle + (cv_1 + dv_2)|1\rangle}$$

$$= \begin{pmatrix} av_1 + bv_2 \\ cv_1 + dv_2 \end{pmatrix} \checkmark$$ (just as before!).

- Arbitrary $d\times d$ matrices:

$$M = \sum_{i,j=0}^{d-1} m_{i,j} |i\rangle\langle j|$$

entry in the i^{th} row and j^{th} column (complex number).

$$\langle k|M|\ell\rangle = \langle k|\left(\sum_{i,j=0}^{d-1} m_{i,j}|i\rangle\langle j|\right)|\ell\rangle$$

$$= \sum_{i,j=0}^{d-1} m_{i,j} \underbrace{\langle k|i\rangle}_{\delta_{k,i}}\underbrace{\langle j|\ell\rangle}_{\delta_{j,\ell}} = m_{k,\ell}$$

$$M_1 = \sum_{i,j=0}^{d-1} m^{(1)}_{i,j}|i\rangle\langle j| \qquad M_2 = \sum_{i,j=0}^{d-1} m^{(2)}_{i,j}|i\rangle\langle j|$$

$$M_1\cdot M_2 = \left(\sum_{i_1,j_1=0}^{d-1} m^{(1)}_{i_1,j_1}|i_1\rangle\langle j_1|\right)\left(\sum_{i_2,j_2=0}^{d-1} m^{(2)}_{i_2,j_2}|i_2\rangle\langle j_2|\right)$$

$$= \sum_{i_1,j_1=0}^{d-1}\sum_{i_2,j_2=0}^{d-1} m^{(1)}_{i_1,j_1} m^{(2)}_{i_2,j_2} |i_1\rangle\underbrace{\langle j_1|i_2\rangle}_{\delta_{j_1,i_2}}\langle j_2| = \sum_{i_1,j_2=0}^{d-1}\left(\sum_{j_1=0}^{d-1} m^{(1)}_{i_1,j_1} m^{(2)}_{j_1,j_2}\right)|i_1\rangle\langle j_2|$$