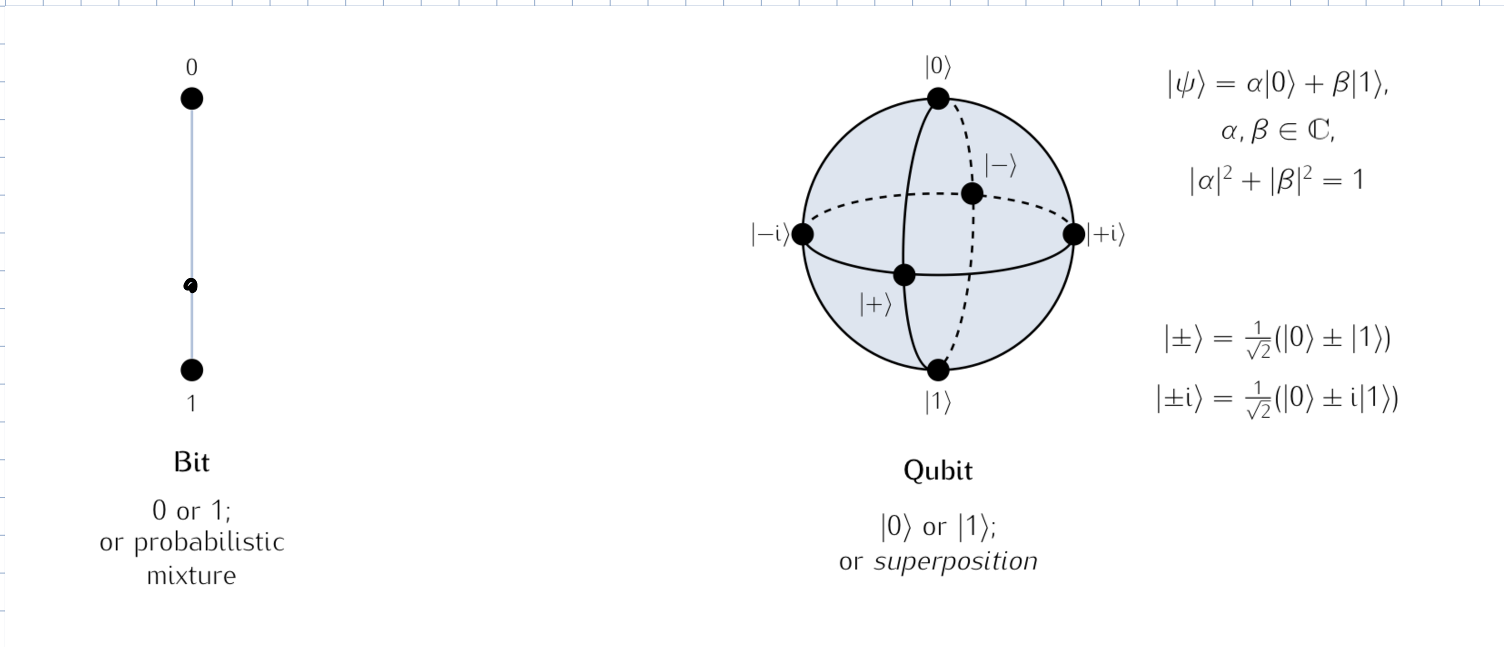
① What is a qubit?



* A bit takes a value either 0 or 1 — or any point in between

→ Probabilistic bit : • Value is 0 with probability p_0.
• Value is 1 with probability p_1.
* $p_0 + p_1 = 1$.

→ So the state of a (probabilistic) bit is just a number b/w 0 and 1!

* The state of a qubit is a point on the sphere — or any point inside the sphere. (We will understand the reasons for this later...).

Bloch sphere

→ We have to describe this using vectors of complex numbers.

→ $|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, $\alpha, \beta \in \mathbb{C}$ (complex numbers).

"ket"

$|\alpha|^2 = \alpha \cdot \bar{\alpha}$: probability of 0

$|\beta|^2 = \beta \cdot \bar{\beta}$: probability of 1.

complex conjugate.

* In quantum mechanics, probabilities are "generated" by complex numbers.

② Complex Numbers

Set of complex numbers.

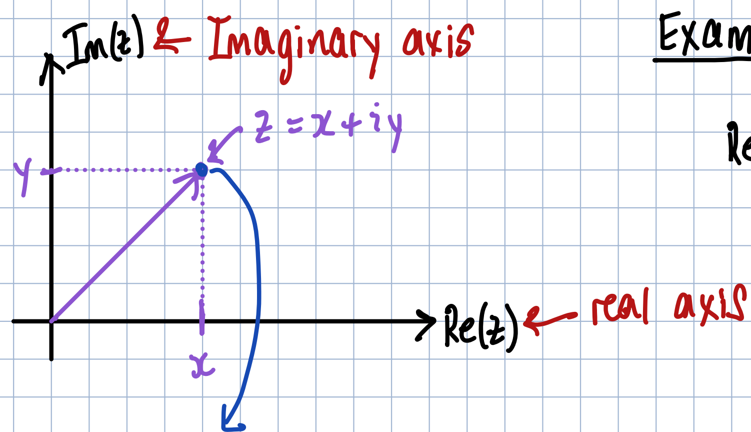
- A complex number is a number $z \in \mathbb{C}$ with a real part and imaginary part

$$z = \underbrace{x}_{\text{Real part}} + \underbrace{y}_{\text{Imaginary part}} i \,, \quad i^2 = -1$$

Example: $z = 3 + 2i$

⊛ Initially conceived in the 1500s for solving polynomial equations

- Can be represented in a 2D plane. (the "complex plane").



Example: $z = 3 + 2i$

$Re(z) = 3$, $Im(z) = 2$.

⊛ We can represent this as a vector: $\vec{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

- Complex conjugate: flip the sign of the imaginary part.

$$z = x + yi \Rightarrow \underbrace{\bar{z}}_{z^*} = x - yi$$

Example: $z = 3 + 2i \Rightarrow \bar{z} = 3 - 2i$.

- Modulus (or magnitude): $|z|^2 = z \cdot \bar{z} = (x + yi)(x - yi)$

$$= x^2 - xyi + yxi + y^2 \underbrace{(i)(-i)}_{=+1} \qquad i^2 = -1$$

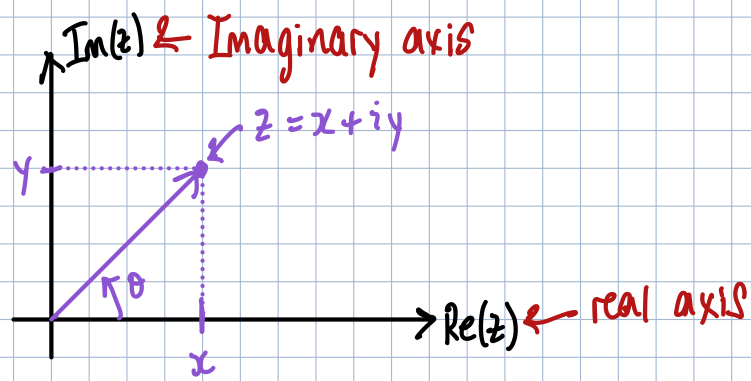
$$= x^2 + y^2$$

$$\Rightarrow |z| = \sqrt{x^2 + y^2}$$

(*) This is just the length of the vector in the complex plane!

also called norm: $\| |v\rangle \| = \sqrt{x^2+y^2}$.

- Polar form:



$z = re^{i\theta}$, $r = \sqrt{x^2+y^2} = |z|$.

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

(*) $e^{i\theta} = \sum_{k=0}^{\infty} \frac{1}{k!}(i\theta)^k = \cos(\theta) + i\sin(\theta)$.

$\Rightarrow x = r\cos(\theta) = Re(z)$, $y = r\sin(\theta) = Im(z)$.

Example: $z = 3+2i \Rightarrow r = \sqrt{3^2+2^2} = \sqrt{9+4} = \sqrt{13}$, $\theta = \tan^{-1}\left(\frac{2}{3}\right) \approx 33.69°$.

- Addition and multiplication:

$z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i \longrightarrow z_1 + z_2 = (x_1+x_2) + (y_1+y_2)i$

$$z_1 + z_2 = x_1 + y_1 i + x_2 + y_2 i = (x_1+x_2) + (y_1+y_2)i$$

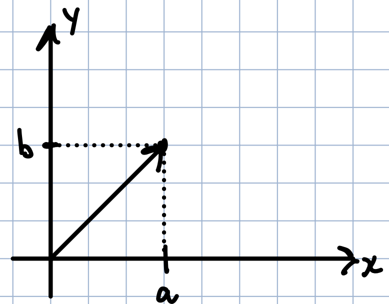
$$\begin{pmatrix} Re(z_1+z_2) = Re(z_1) + Re(z_2) \\ Im(z_1+z_2) = Im(z_1) + Im(z_2). \end{pmatrix}$$

$$z_1 \cdot z_2 = (x_1 + y_1 i)\cdot(x_2 + y_2 i) = x_1x_2 + x_1y_2 i + y_1x_2 i + y_1 \cdot y_2 \underbrace{(i \cdot i)}_{=-1}$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

$$z_1 = r_1 e^{i\theta_1}, \; z_2 = r_2 e^{i\theta_2} \Rightarrow z_1 z_2 = r_1 \cdot r_2 e^{i\theta_1} e^{i\theta_2} = r_1 \cdot r_2 e^{i(\theta_1+\theta_2)}$$

③ Complex Vector Spaces

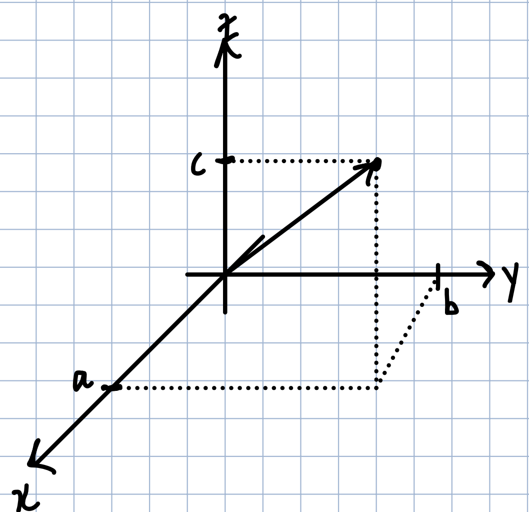
- Recall 2D and 3D vectors from linear algebra



$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$, $a, b \in \mathbb{R}$ → real numbers.

• We write $\vec{v} \in \mathbb{R}^2$ → all 2D vectors of real numbers.

(Note: this is basically the same as the complex plane!)



• In 3D, vectors have three components.

$$\vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} ; \ a, b, c \in \mathbb{R}.$$

• We write $\vec{v} \in \mathbb{R}^3$.

- For any dimension $d \in \{2, 3, \dots\}$, we have $\vec{v} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix}$, $a_k \in \mathbb{R}$.

and we write $\vec{v} \in \mathbb{R}^d$. The norm (magnitude) of $\vec{v}$ is $\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_d^2}$.

- We add (and subtract) vectors component-wise

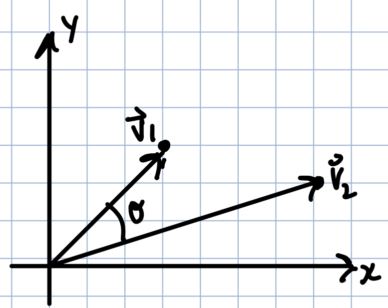
$$\vec{v}_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{pmatrix} \quad \Rightarrow \quad \vec{v}_1 + \vec{v}_2 = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_d + b_d \end{pmatrix}.$$

- We can take the <u>dot product</u> of two vectors.

$$\vec{v}_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{pmatrix} \Rightarrow \vec{v}_1 \cdot \vec{v}_2 = a_1 b_1 + a_2 b_2 + \cdots + a_d b_d. = \sum_{k=1}^{d} a_k b_k$$

⊛ <u>Observe</u>: $\vec{v}_1 \cdot \vec{v}_1 = a_1^2 + a_2^2 + \cdots + a_d^2 = \|\vec{v}\|^2$.

Geometric interpretation in 2D:



$$\vec{v}_1 \cdot \vec{v}_2 = \|\vec{v}_1\| \cdot \|\vec{v}_2\| \cdot \cos\theta$$

⊛ So the dot product tells us how much the two vectors overlap.

- <u>Complex vectors</u> are similar!

$$\vec{v} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix}$$ → but now each $a_k \in \mathbb{C}$! (Each a_k is a complex number.)

$a_k = x_k + y_k i$

⊛ <u>We write $\vec{v} \in \mathbb{C}^d$.</u>

- Addition as before: $\vec{v}_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{pmatrix} \Rightarrow \vec{v}_1 + \vec{v}_2 = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_d + b_d \end{pmatrix}$.

- But dot product changes! And we call it "<u>inner product</u>" instead:

$$\vec{v}_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \quad \vec{v}_2 = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{pmatrix} \Rightarrow \langle \vec{v}_1, \vec{v}_2 \rangle = \bar{a}_1 b_1 + \bar{a}_2 b_2 + \cdots + \bar{a}_d b_d.$$

- Conjugate transpose of a vector:

$$\vec{v}^T = \begin{pmatrix} a_1 & a_2 & \cdots & a_d \end{pmatrix}$$

$$\vec{v}_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \rightarrow \vec{v}_1^{\dagger} = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_d \end{pmatrix}$$

⊛ Observe: $\vec{v}_1^{\dagger} \vec{v}_2 = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_d \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_d \end{pmatrix} = \bar{a}_1 b_1 + \bar{a}_2 b_2 + \cdots + \bar{a}_d b_d = \langle \vec{v}_1, \vec{v}_2 \rangle.$

We will use this fact all the time in quantum computing!

⊛ Norm of a complex vector is $\|\vec{v}\| = \sqrt{\underbrace{|a_1|^2}_{a_1 \cdot \bar{a}_1} + |a_2|^2 + \cdots + |a_d|^2} = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

- <u>Basis of a vector space.</u>

  - We can write a vector as

$$\vec{v}_1 = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \\ \vdots \\ 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ a_d \end{pmatrix}$$

$$= a_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{e}_1} + a_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{\vec{e}_2} + \cdots + a_d \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{\vec{e}_d}$$

$$= \sum_{k=1}^{d} a_k \vec{e}_k$$

The "standard" basis

⊛ $\{\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_d\}$ are <u>basis vectors</u>. Every vector can be written as a <u>linear combination</u> of these basis vectors.

⊛ These basis vectors are <u>orthonormal</u>

Two parts to this term:

$d=3 \rightarrow \vec{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \vec{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$\vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

1. **Orthogonal**: $\langle \vec{e}_i, \vec{e}_j \rangle = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$\equiv \delta_{i,j} \rightarrow$ "Kronecker delta"

2. **Normalised**: $\|\vec{e}_k\| = \sqrt{\langle \vec{e}_k, \vec{e}_k \rangle} = 1$ for all k.

$\langle \vec{e}_1, \vec{e}_2 \rangle = (1\ 0\ 0)\begin{pmatrix}0\\1\\0\end{pmatrix} = 0$

$\langle \vec{e}_1, \vec{e}_1 \rangle = (1\ 0\ 0)\begin{pmatrix}1\\0\\0\end{pmatrix} = 1$

⊛ We can extract the components of a vector in this basis using the inner product:

$$\langle \vec{e}_k, \vec{v} \rangle = \left\langle \vec{e}_k, \sum_{l=1}^{d} a_l \vec{e}_l \right\rangle = \sum_{l=1}^{d} \underbrace{\langle \vec{e}_k, \vec{e}_l \rangle}_{\delta_{k,l}} a_l = a_k.$$

$$\langle \vec{e}_1, a_1\vec{e}_1 + a_2\vec{e}_2 \rangle = a_1 \underbrace{\langle \vec{e}_1, \vec{e}_1 \rangle}_{=1} + a_2 \underbrace{\langle \vec{e}_1, \vec{e}_2 \rangle}_{=0} = a_1$$

– **Bra-ket notation**: very important, used throughout quantum information and quantum computing.

$|v_1\rangle = \sum_{k=1}^{d} a_k |e_k\rangle$, $|e_1\rangle = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}$

$|v_1\rangle$

$\cancel{\vec{v}_1} = \begin{pmatrix}a_1\\a_2\\\vdots\\a_d\end{pmatrix} \rightarrow \vec{v}_1^\dagger = (\bar{a}_1\ \bar{a}_2\ \cdots\ \bar{a}_d)$

$\rightarrow$ relabel as $\langle v_1|$ $\rightarrow$ call it "bra"

relabel as $|v_1\rangle \rightarrow$ call it a "ket".

• Then the inner product is $\langle \vec{v}_1, \vec{v}_2 \rangle = \vec{v}_1^\dagger v_2 = \langle v_1 | v_2 \rangle$

"bra-ket".

$\delta_{k,k'}$

- We write the basis vectors as $|e_k\rangle$

$$\left(\sum_{k=1}^{d} \bar{a}_k \langle e_k|\right)\left(\sum_{k'=1}^{d} a_{k'} |e_{k'}\rangle\right) = \sum_{k,k'=1}^{d} \bar{a}_k a_{k'} \langle e_k|e_{k'}\rangle$$

$$\rightarrow |v\rangle = \sum_{k=1}^{d} a_k |e_k\rangle \rightarrow \langle v| = \sum_{k=1}^{d} \bar{a}_k \langle e_k| . \rightarrow \langle v|v\rangle = \sum_{k=1}^{d} \bar{a}_k \cdot a_k = \underline{\sum_{k=1}^{d} |a_k|^2 = \| |v\rangle \|^2}.$$

⊛ With this abstract notation, we do not have to explicitly write column vectors! (Helpful for large vectors — for n qubits the size of the vectors is 2^n!)