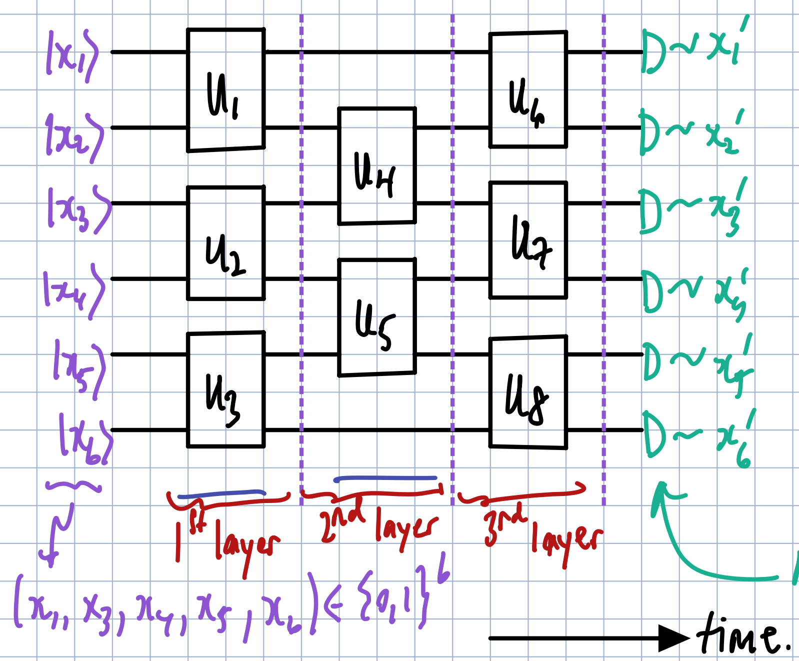
① Recap: Quantum Circuits

(*) The circuit elements/gates are **unitaries**.
$(U : U^\dagger U = U U^\dagger = \mathbb{1})$.



Measurement/read-out

(a) Pauli gates:

$-\boxed{X}- \;\rightarrow\; X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \big(X|0\rangle = |1\rangle,\; X|1\rangle = |0\rangle\big)$

$-\boxed{Y}- \;\rightarrow\; Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \big(Y|0\rangle = i|1\rangle,\; Y|1\rangle = -i|0\rangle\big)$

$-\boxed{Z}- \;\rightarrow\; Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \big(Z|0\rangle = |0\rangle,\; Z|1\rangle = -|1\rangle\big)$

(b) Hadamard gate:

$-\boxed{H}- \;\rightarrow\; H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

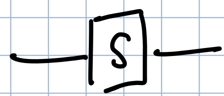
$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$

$H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big).$

$X|+\rangle = |+\rangle$

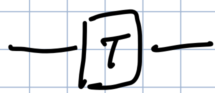
$X|-\rangle = -|-\rangle$

(c) Phase gate :



$\rightarrow \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$

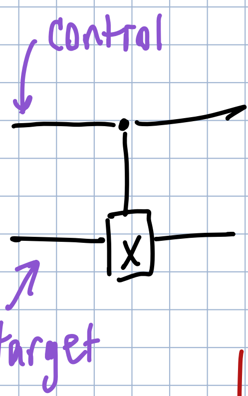
(d) T - gate :



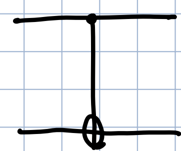
$\rightarrow \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$

(e) CNOT / CX gate :

"controlled-X"



or



$\circledast \; |a,b\rangle \mapsto |a, a\oplus b\rangle$

$$\text{CNOT} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$|0\rangle|0\rangle \longmapsto |0\rangle|0\rangle$

$|0\rangle|1\rangle \longmapsto |0\rangle|1\rangle$

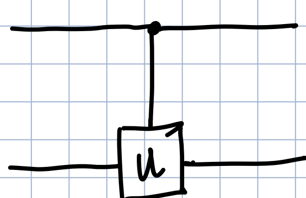
$|1\rangle|0\rangle \longmapsto |1\rangle X|0\rangle = |1\rangle|1\rangle$

$|1\rangle|1\rangle \longmapsto |1\rangle X|1\rangle = |1\rangle|0\rangle$

B/c of linearity, this determines the action on <u>any</u> state!

$|\psi\rangle = a|0,0\rangle + b|0,1\rangle + c|1,0\rangle + d|1,1\rangle \mapsto a|0,0\rangle + b|0,1\rangle + c|1,1\rangle + d|1,0\rangle$

(f) Controlled Unitary



$|0\rangle|0\rangle \longmapsto |0\rangle|0\rangle$

$|0\rangle|1\rangle \longmapsto |0\rangle|1\rangle$

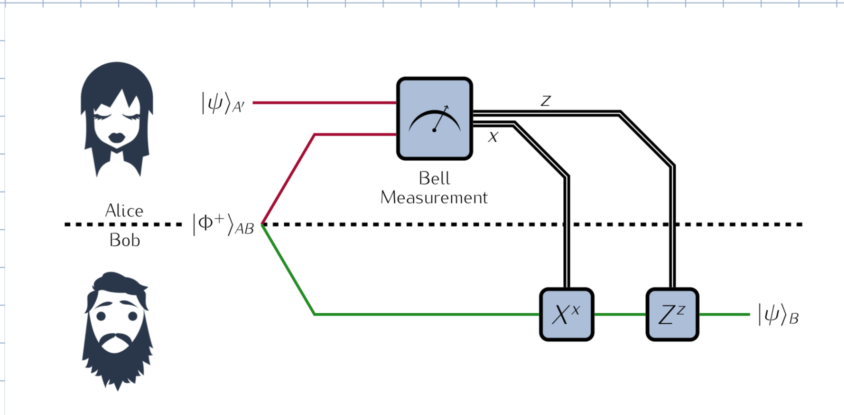
$|1\rangle|0\rangle \longmapsto |1\rangle U|0\rangle$

$|1\rangle|1\rangle \longmapsto |1\rangle U|1\rangle$

$$cU = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & U \end{pmatrix} = |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes U.$$

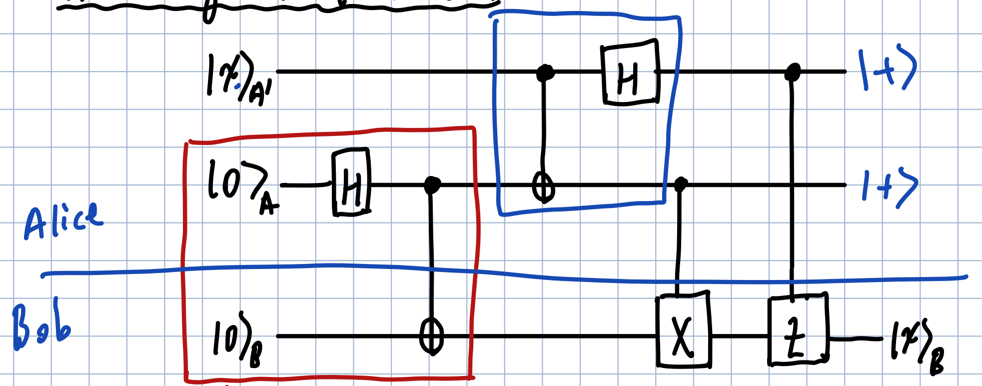
② Examples

(a) Teleportation



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle.$

As a quantum circuit



$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|0\rangle_A|0\rangle_B \mapsto |+\rangle_A|0\rangle_B = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|0\rangle_B) \mapsto \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) = |\Phi^+\rangle$

$|\psi\rangle_{A'}|0\rangle_A|0\rangle_B \longmapsto |\psi\rangle_{A'} \otimes |\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|\psi\rangle_{A'}|0\rangle_A|0\rangle_B + |\psi\rangle_{A'}|1\rangle_A|1\rangle_B)$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$$= \frac{1}{\sqrt{2}}\big(\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \beta|1\rangle_{A'}|0\rangle_A|0\rangle_B + \alpha|0\rangle_{A'}|1\rangle_A|1\rangle_B + \beta|1\rangle_{A'}|1\rangle_A|1\rangle_B\big).$$

$$\xmapsto[CNOT_{A'A}]{} \frac{1}{\sqrt{2}}\big(\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \beta|1\rangle_{A'}|1\rangle_A|0\rangle_B + \alpha|0\rangle_{A'}|1\rangle_A|1\rangle_B + \beta|1\rangle_{A'}|0\rangle_A|1\rangle_B\big).$$

$$
\begin{aligned}
\xmapsto[H_{A'}]{} \frac{1}{2}\Big(&\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|0\rangle_A|0\rangle_B \\
&+ \beta|0\rangle_{A'}|1\rangle_A|0\rangle_B - \beta|1\rangle_{A'}|1\rangle_A|0\rangle_B \\
&+ \alpha|0\rangle_{A'}|1\rangle_A|1\rangle_B + \alpha|1\rangle_{A'}|1\rangle_A|1\rangle_B \\
&+ \beta|0\rangle_{A'}|0\rangle_A|1\rangle_B - \beta|1\rangle_{A'}|0\rangle_A|1\rangle_B\Big).
\end{aligned}
$$

$$
\begin{aligned}
\xmapsto[CNOT_{AB}]{} \frac{1}{2}\Big(&\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|0\rangle_A|0\rangle_B \\
&+ \beta|0\rangle_{A'}|1\rangle_A|1\rangle_B - \beta|1\rangle_{A'}|1\rangle_A|1\rangle_B \\
&+ \alpha|0\rangle_{A'}|1\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|1\rangle_A|0\rangle_B \\
&+ \beta|0\rangle_{A'}|0\rangle_A|1\rangle_B - \beta|1\rangle_{A'}|0\rangle_A|1\rangle_B\Big).
\end{aligned}
$$

$$
\begin{aligned}
\xmapsto[CZ_{A'B}]{} \frac{1}{2}\Big(&\alpha|0\rangle_{A'}|0\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|0\rangle_A|0\rangle_B \\
&+ \beta|0\rangle_{A'}|1\rangle_A|1\rangle_B + \beta|1\rangle_{A'}|1\rangle_A|1\rangle_B \\
&+ \alpha|0\rangle_{A'}|1\rangle_A|0\rangle_B + \alpha|1\rangle_{A'}|1\rangle_A|0\rangle_B \\
&+ \beta|0\rangle_{A'}|0\rangle_A|1\rangle_B + \beta|1\rangle_{A'}|0\rangle_A|1\rangle_B\Big).
\end{aligned}
$$

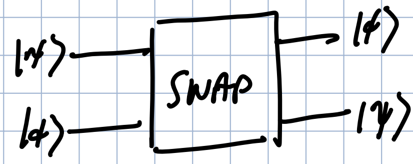
$$
\begin{aligned}
= \frac{1}{2}\Big(&|0\rangle_{A'}|0\rangle_A\big(\alpha|0\rangle_B + \beta|1\rangle_B\big) \\
&+ |0\rangle_{A'}|1\rangle_A\big(\alpha|0\rangle_B + \beta|1\rangle_B\big) \\
&+ |1\rangle_{A'}|0\rangle_A\big(\alpha|0\rangle_B + \beta|1\rangle_B\big) \\
&+ |1\rangle_{A'}|1\rangle_A\big(\alpha|0\rangle_B + \beta|1\rangle_B\big)\Big)
\end{aligned}
$$

$$
= \frac{1}{2}\left(|0\rangle_{A'}|0\rangle_A + |0\rangle_{A'}|1\rangle_A + |1\rangle_{A'}|0\rangle_A + |1\rangle_{A'}|1\rangle_A\right)|\psi\rangle_B
$$

$$
= \underline{\underline{|+\rangle_{A'}|+\rangle_A|\psi\rangle_B}}
$$

(b) The "SWAP test" (How do we estimate the inner product of two states on a quantum computer?)

- The SWAP gate:



$|0\rangle|0\rangle \mapsto |0\rangle|0\rangle$

$|0\rangle|1\rangle \mapsto |1\rangle|0\rangle$

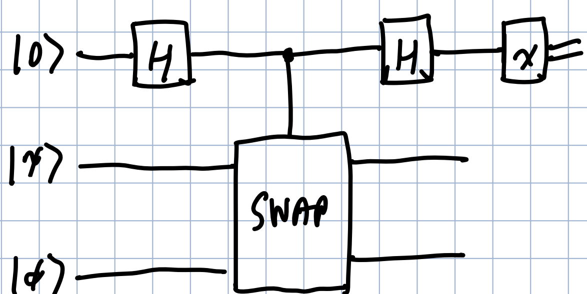
$|1\rangle|0\rangle \mapsto |0\rangle|1\rangle$

$|1\rangle|1\rangle \mapsto |1\rangle|1\rangle$

As a matrix:

$$\begin{array}{c|c} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

- Circuit for the SWAP test



$$|\psi_{init}\rangle = |0\rangle|\psi\rangle|\phi\rangle \overset{H}{\mapsto} \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle|\phi\rangle + |1\rangle|\psi\rangle|\phi\rangle\right)$$

$$\overset{cSWAP}{\mapsto} \frac{1}{\sqrt{2}}\left(|0\rangle|\psi\rangle|\phi\rangle + |1\rangle|\phi\rangle|\psi\rangle\right)$$

$$\overset{H}{\mapsto} \frac{1}{\sqrt{2}}\left(|+\rangle|\psi\rangle|\phi\rangle + |-\rangle|\phi\rangle|\psi\rangle\right)$$

$$|\psi_{final}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \frac{1}{\sqrt{2}}\left(|\psi\rangle|\phi\rangle + |\phi\rangle|\psi\rangle\right) + |1\rangle \frac{1}{\sqrt{2}}\left(|\psi\rangle|\phi\rangle - |\phi\rangle|\psi\rangle\right)\right)$$

Now measure: What is the probability of getting 0? (Recall partial measurements)

$$\Pr[0] = \mathrm{Tr}\left[\left(|0\rangle\langle 0| \otimes \mathbb{1} \otimes \mathbb{1}\right)|\psi_{final}\rangle\langle\psi_{final}|\right].$$

$$= \frac{1}{4}\mathrm{Tr}\left[\left(|\gamma\rangle|\phi\rangle + |\phi\rangle|\gamma\rangle\right)\left(\langle\gamma|\langle\phi| + \langle\phi|\langle\gamma|\right)\right]$$

$$= \frac{1}{4}\left(\langle\gamma|\langle\phi| + \langle\phi|\langle\gamma|\right)\left(|\gamma\rangle|\phi\rangle + |\phi\rangle|\gamma\rangle\right)$$

$$= \frac{1}{4}\Big(\underbrace{\langle\gamma|\gamma\rangle\langle\phi|\phi\rangle}_{=1} + \underbrace{\langle\gamma|\phi\rangle\langle\phi|\gamma\rangle + \langle\phi|\gamma\rangle\langle\gamma|\phi\rangle}_{=2|\langle\gamma|\phi\rangle|^2} + \underbrace{\langle\phi|\phi\rangle\langle\gamma|\gamma\rangle}_{=1}\Big)$$

$$= \frac{1}{2}\left(1 + |\langle\gamma|\phi\rangle|^2\right)$$

$$\Pr[1] = \frac{1}{2}\left(1 - |\langle\gamma|\phi\rangle|^2\right)$$

⊛ This defines a random variable X: $\Pr[X=\pm 1] = \frac{1}{2}\left(1 \pm |\langle\gamma|\phi\rangle|^2\right)$.

⊛ Now do this many times

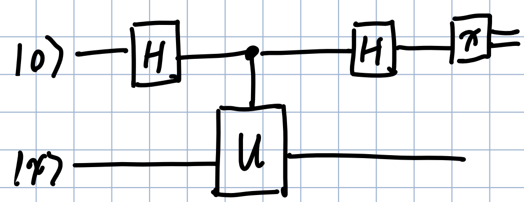
- Each time you get outcome "0" $\rightarrow$ record $x_i = 1$
- Each time you get outcome "1" $\rightarrow$ record $x_i = -1$
- Take the sample mean: $\hat{X}_N = \frac{1}{N}\sum_{i=1}^{N} x_i$ (This is an unbiased estimator of X.)
- As $N \to \infty$, $\hat{X}_N \to \mathbb{E}[X]$ (law of large numbers). $\mathbb{E}[\hat{X}_N] = \mathbb{E}[X]$
- $\mathbb{E}[X] = \Pr[X=1]\cdot 1 + \Pr[X=-1]\cdot(-1)$ $\qquad \mathbb{E}[X] = \sum_x \Pr[X=x]\cdot x$

$$= \frac{1}{2}\left(1 + |\langle\gamma|\phi\rangle|^2\right) - \frac{1}{2}\left(1 - |\langle\gamma|\phi\rangle|^2\right)$$

$$= |\langle\gamma|\phi\rangle|^2 \checkmark$$

(c) <u>Hadamard Test</u>: Estimating $\langle\psi|U|\psi\rangle$, where U is a unitary and $|\psi\rangle$ is a state vector.

$U = e^{-iHt}$



$$|\psi_{init}\rangle = |0\rangle|\psi\rangle \longmapsto |+\rangle|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle\big) \longmapsto \frac{1}{\sqrt{2}}\big(|0\rangle|\psi\rangle + |1\rangle U|\psi\rangle\big)$$

$$\longmapsto \frac{1}{\sqrt{2}}\big(|+\rangle|\psi\rangle + |-\rangle U|\psi\rangle\big) = \frac{1}{\sqrt{2}}\Big(|0\rangle\frac{1}{\sqrt{2}}\big(|\psi\rangle + U|\psi\rangle\big) + |1\rangle\frac{1}{\sqrt{2}}\big(|\psi\rangle - U|\psi\rangle\big)\Big) = |\psi_{final}\rangle.$$

$$\Pr[0] = \mathrm{Tr}\big[(|0\rangle\langle 0| \otimes \mathbb{1})|\psi_{final}\rangle\langle\psi_{final}|\big]$$

$$= \frac{1}{4}\Big(\big(\langle\psi| + \langle\psi|U^\dagger\big)\big(|\psi\rangle + U|\psi\rangle\big)\Big)$$

$$= \frac{1}{4}\Big(\langle\psi|\psi\rangle + \langle\psi|U|\psi\rangle + \underbrace{\langle\psi|U^\dagger|\psi\rangle}_{=\overline{\langle\psi|U|\psi\rangle}} + \langle\psi|\underbrace{U^\dagger U}_{=\mathbb{1}}|\psi\rangle\Big)$$

$$= \frac{1}{2}\big(1 + \mathrm{Re}(\langle\psi|U|\psi\rangle)\big)$$

$z = x + iy$

$\bar{z} = x - iy$

$z + \bar{z} = 2x = 2\,\mathrm{Re}(z)$

$z - \bar{z} = 2iy = 2i\,\mathrm{Im}(z)$

$$\Pr[1] = \frac{1}{2}\big(1 - \mathrm{Re}(\langle\psi|U|\psi\rangle)\big)$$

⊛ Now do the same as before!

⊛ Do the circuit many times.

- Each time you get outcome "0" → record $x_i = 1$
- Each time you get outcome "1" → record $x_i = -1$

⊛ This defines a <u>random variable</u> X: $\Pr[X = \pm 1] = \frac{1}{2}\big(1 \pm \mathrm{Re}(\langle\psi|U|\psi\rangle)\big)$

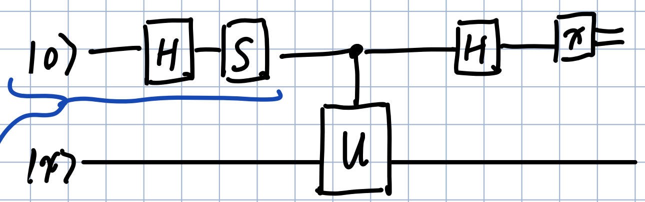
- Take the sample mean: $\hat{X}_N = \frac{1}{N}\sum_{i=1}^{N} x_i$ (This is an <u>unbiased estimator</u> of X.)
- As $N \to \infty$, $\hat{X}_N \to \mathbb{E}[X]$ (law of large numbers).

$$\bullet\ \mathbb{E}[X] = \Pr[X=1]\cdot 1 + \Pr[X=-1]\cdot(-1)$$

$$= \frac{1}{2}\left(1+\mathrm{Re}(\langle\psi|U|\psi\rangle)\right) - \frac{1}{2}\left(1-\mathrm{Re}(\langle\psi|U|\psi\rangle)\right)$$

$$= \mathrm{Re}(\langle\psi|U|\psi\rangle).$$

⊛ For the imaginary part → include the S gate!



$$\Pr[0] = \frac{1}{2}\left(1+\mathrm{Im}(\langle\psi|U|\psi\rangle)\right),\quad \Pr[1] = \frac{1}{2}\left(1-\mathrm{Im}(\langle\psi|U|\psi\rangle)\right).$$

$$|0\rangle \xmapsto{H} |+\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle+|1\rangle\right) \xmapsto{S} \frac{1}{\sqrt{2}}\left(|0\rangle+\underline{i}|1\rangle\right)$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$S|0\rangle = |0\rangle$$
$$S|1\rangle = i|1\rangle$$