① Multi-qubit Entanglement.

(a) Recall the Bell state $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\sum_{k=0}^{1} |k,k\rangle \longrightarrow$ Entangled state of $\underline{\underline{\text{two}}}$ qubits.

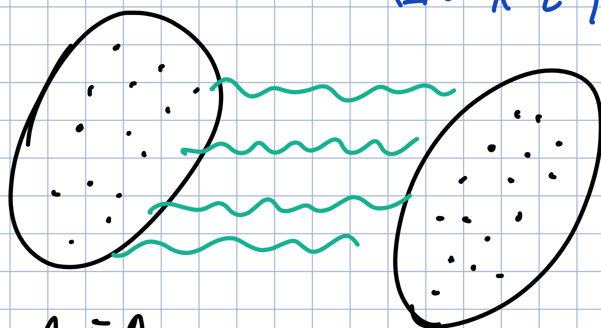
$\{|0\rangle, |1\rangle\}$

$\hookrightarrow \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$ $\quad \frac{1}{\sqrt{2}}(|0,1\rangle + |1,0\rangle)$

$\frac{1}{\sqrt{2}}(|0,0\rangle - |1,1\rangle)$ $\quad \frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$

(b) Entangled states of multiple qubits

$(\mathbb{1} \otimes X^x Z^z)|\Phi^+\rangle, \; x,z \in \{0,1\}$



$A_1, A_2, \ldots, A_n \equiv A_{1:n}$ (n qubits)

$B_1, B_2, \ldots, B_n \equiv B_{1:n}$ (n qubits).

$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$

$|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

$|0,0\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$

$|0,1\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

• Recall: Basis for $(\mathbb{C}^2)^{\otimes n}$ is $\{|\vec{k}\rangle : \vec{k} \in \{0,1\}^n\} \rightarrow 2^n$

Example: For $n=2$: $\{|0,0\rangle, |0,1\rangle, |1,0\rangle, |1,1\rangle\}$

For $n=3$: $\{|0,0,0\rangle, |0,0,1\rangle, |0,1,0\rangle, |0,1,1\rangle$
$|1,0,0\rangle, |1,0,1\rangle, |1,1,0\rangle, |1,1,1\rangle\}$

• Bell state of two n-qubit systems: $|\Phi^+\rangle_{A_{1:n}B_{1:n}} = \frac{1}{\sqrt{2^n}} \sum_{\vec{k}\in\{0,1\}^n} |\vec{k},\vec{k}\rangle_{A_{1:n}B_{1:n}}$

$\vec{k} = (k_1, k_2, \ldots, k_n)$, $k_i \in \{0,1\}$

Observe: $|\vec{k},\vec{k}\rangle_{A_{1:n}B_{1:n}} = |k_1, k_2, \ldots, k_n\rangle_{A_{1:n}} \otimes |k_1, k_2, \ldots, k_n\rangle_{B_{1:n}}$

$\xrightarrow{\text{reordering}} |k_1,k_1\rangle_{A_1B_1} \otimes |k_2,k_2\rangle_{A_2B_2} \otimes \cdots \otimes |k_n,k_n\rangle_{A_nB_n}$

$\Rightarrow |\Phi^+\rangle_{A_{1:n}B_{1:n}} \xrightarrow{\text{reordering}} |\Phi^+\rangle_{A_1B_1} \otimes |\Phi^+\rangle_{A_2B_2} \otimes \cdots \otimes |\Phi^+\rangle_{A_nB_n}$

$$\frac{1}{\sqrt{2^n}}\sum_{\vec{k}\in\{0,1\}^n} |\vec{k},\vec{k}\rangle_{A_{1:n}B_{1:n}} = \frac{1}{\sqrt{2^n}}\sum_{k_1,k_2,\dots,k_n} |k_1,k_2,\dots,k_n\rangle_{A_{1:n}} \otimes |k_1,k_2,\dots,k_n\rangle_{B_{1:n}}$$

$$= \frac{1}{\sqrt{2^n}}\sum_{k_1,k_2,\dots,k_n} |k_1,k_1\rangle_{A_1B_1} \otimes |k_2,k_2\rangle_{A_2B_2} \otimes \dots \otimes |k_n,k_n\rangle_{A_nB_n}$$

$$= \underbrace{\left(\frac{1}{\sqrt{2}}\sum_{k_1} |k_1,k_1\rangle\right)}_{|\Phi^+\rangle_{A_1B_1}} \otimes \underbrace{\left(\frac{1}{\sqrt{2}}\sum_{k_2} |k_2,k_2\rangle\right)}_{|\Phi^+\rangle_{A_2B_2}} \otimes \dots \otimes \underbrace{\left(\frac{1}{\sqrt{2}}\sum_{k_n} |k_n,k_n\rangle\right)}_{|\Phi^+\rangle_{A_nB_n}}$$

- Entire Bell basis (of $2n$ qubits): $|\Phi_{\vec{z},\vec{x}}\rangle_{A_{1:n}B_{1:n}} = (Z^{\vec{z}}X^{\vec{x}} \otimes \mathbb{1})|\Phi^+\rangle_{A_{1:n}B_{1:n}}$

$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$Z^{101} = Z \otimes \mathbb{1} \otimes Z$

$\vec{z} = (z_1, z_2, \ldots, z_n) \Rightarrow Z^{\vec{z}} \equiv Z^{z_1} \otimes Z^{z_2} \otimes \cdots \otimes Z^{z_n}$

$\vec{x} = (x_1, x_2, \ldots, x_n) \Rightarrow X^{\vec{x}} \equiv X^{x_1} \otimes X^{x_2} \otimes \cdots \otimes X^{x_n}$

✱ $\{|\Phi_{\vec{z},\vec{x}}\rangle : \vec{z}, \vec{x} \in \{0,1\}^n\}$ is an ONB for $(\mathbb{C}^2)^{\otimes n} \otimes (\mathbb{C}^2)^{\otimes n}$

- GHZ states: $|GHZ_n\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}\right) \rightarrow n=3 \Rightarrow \frac{1}{\sqrt{2}}\left(|0,0,0\rangle + |1,1,1\rangle\right)$

Basis for $(\mathbb{C}^2)^{\otimes n}$: $|GHZ_{z,\vec{x}}\rangle = (Z^{z} \otimes X^{\vec{x}})|GHZ_n\rangle$, $z \in \{0,1\}$, $\vec{x} = \{0,1\}^{n-1}$.

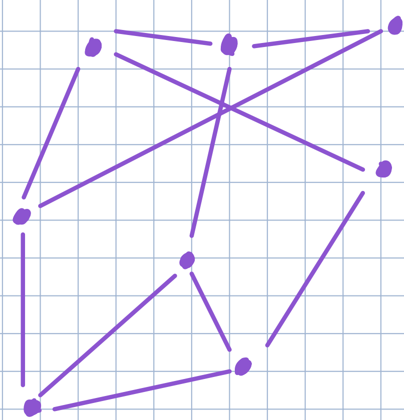
(✱ Useful in quantum sensing and cryptography.)

- Graph States: (A graph $G = (V, E)$)

edges.

nodes/vertices

(✱ Useful in quantum computing!)



Recipe:

1. At every vertex, put the $|+\rangle$ state. $\left(|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)$

2. For every edge, apply the controlled-Z gate

$e^{i\varphi}$

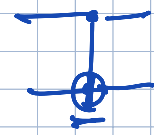
$$CZ = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

$CZ|0,0\rangle = |0,0\rangle$

$CZ|0,1\rangle = |0,1\rangle$

$CZ|1,0\rangle = |1,0\rangle$

$CZ|1,1\rangle = |1\rangle Z|1\rangle = -|1,1\rangle.$



$Z|0\rangle = |0\rangle$ $\quad$ $Z|1\rangle = -|1\rangle$

General formula: $|+\rangle^{\otimes n} = \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right)^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{\vec{x} \in \{0,1\}^n} |\vec{x}\rangle$

$CZ|x_1, x_2\rangle = (-1)^{x_1 x_2} |x_1, x_2\rangle$

$n=2$: $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$$\Rightarrow |G\rangle = \frac{1}{\sqrt{2^n}} \sum_{\vec{x} \in \{0,1\}^n} (-1)^{\alpha_{\vec{x}}} |\vec{x}\rangle, \quad \alpha_{\vec{x}} = \sum_{\{i,j\} \in E} x_i x_j$$

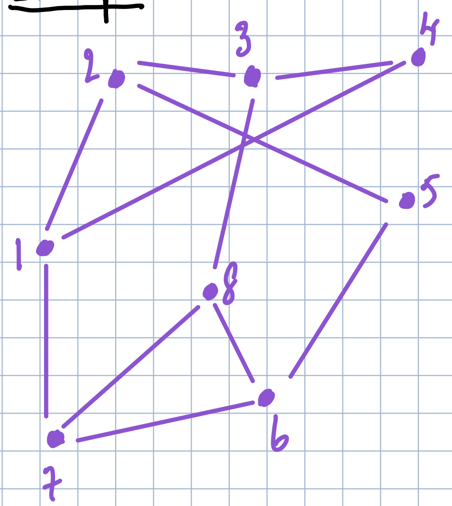
$\hookrightarrow \alpha_{\vec{x}} = x_1 x_4 + x_1 x_2 + x_1 x_7 + \cdots$

$$= \frac{1}{\sqrt{2^n}} \sum_{\vec{x} \in \{0,1\}^n} (-1)^{\frac{1}{2}\vec{x}^T A_G \vec{x}} |\vec{x}\rangle$$

Adjacency matrix of the graph

$$(A_G)_{ij} = \begin{cases} 1 & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise.} \end{cases}$$

Example:



$A_G =$

	1	2	3	4	5	6	7	8
1		1		1			1	
2	1		1		1			
3		1		1				1
4	1		1					
5		1				1		
6					1		1	1
7	1					1		1
8			1			1	1	

② Determining Entanglement.

⊛ Given a state vector $|\psi\rangle$, how to determine if it is entangled or not?
Precisely: $\overset{?}{\exists}\ |\phi_1\rangle, |\phi_2\rangle$ s.t. $|\psi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle$?

(a) $|\psi\rangle_{AB} \in \mathbb{C}^d \otimes \mathbb{C}^d \rightarrow |\psi\rangle_{AB} = \sum_{i,j=0}^{d-1} \underbrace{x_{i,j}}_{\text{two-index quantity} \leftrightarrow \text{matrix!}} |i\rangle \otimes |j\rangle$ $\qquad \langle i,j|\psi\rangle = x_{i,j}$

Lemma

Let $X = \sum_{i,j=0}^{d-1} x_{i,j} |i\rangle\langle j| \in L(\mathbb{C}^d) \Rightarrow \mathrm{vec}(X) = \sum_{i,j=0}^{d-1} x_{i,j}\, \mathrm{vec}(|i\rangle\langle j|) = \sum_{i,j=0}^{d-1} x_{i,j}|ij\rangle$
$= |\psi\rangle!$

$\Rightarrow$ ⊛ For every $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$, $\exists\ X \in L(\mathbb{C}^d)$ such that $|\psi\rangle = \mathrm{vec}(X) = (X \otimes \mathbb{1})|\Gamma\rangle$.

⊛ For every $|\psi\rangle \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$, $\exists\ X \in L(\mathbb{C}^{d_2}, \mathbb{C}^{d_1})$ s.t. $|\psi\rangle = (X \otimes \mathbb{1})|\Gamma_{d_2}\rangle$

(b) Every $X \in L(\mathbb{C}^d)$ has a singular-value decomposition:

$X = U \Sigma V^\dagger$ — matrix of singular values

unitaries

$\Sigma = \begin{pmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & s_r \end{pmatrix}$

singular values, $s_k > 0\ \forall k$.
(unique up to ordering)

In bra-ket notation: $X = \sum_{k=1}^{r} s_k |e_k\rangle\langle f_k|$

rank $\equiv$ # of singular values $\rightarrow$ unique

$\{|e_k\rangle\}_k, \{|f_k\rangle\}_k$ are orthonormal (left and right singular vectors).

$$\Rightarrow |\psi\rangle = \mathrm{vec}(X) = (\mathbb{1}\otimes X)|\Gamma\rangle = \sum_{k=1}^{r} s_k\, \mathrm{vec}(|e_k\rangle\langle f_k|) = \sum_{k=1}^{r} \underbrace{s_k}_{\text{Schmidt coefficients.}} \overline{|e_k\rangle}\otimes|f_k\rangle$$

$$\left(\text{Normalisation: } \langle\psi|\psi\rangle = 1 \Rightarrow \sum_{k=1}^{r} s_k^2 = 1\right)$$

$$\langle\psi|\psi\rangle = \left(\sum_{k=1}^{r} s_k \overline{\langle e_k|}\otimes\langle f_k|\right)\left(\sum_{k'=1}^{r} s_{k'} \overline{|e_{k'}\rangle}\otimes|f_{k'}\rangle\right) = \sum_{k,k'=1}^{r} s_k s_{k'}\,\delta_{k,k'} = \sum_{k=1}^{r} s_k^2 = 1$$

- If $r=1$, then $|\psi\rangle = |e_1\rangle\otimes|f_1\rangle \Rightarrow$ not entangled!
- If $r>1$, then entangled! $r \equiv$ Schmidt rank.

(c) Observe: $|\psi\rangle_{AB} = \sum_{k=1}^{r} s_k \overline{|e_k\rangle}_A \otimes |f_k\rangle_B$

$$\Rightarrow \mathrm{Tr}_B\left[|\psi\rangle\langle\psi|_{AB}\right] = \mathrm{Tr}_B\left[\sum_{k,k'=1}^{r} s_k s_{k'} \overline{|e_k\rangle}\overline{\langle e_{k'}|}_A \otimes |f_k\rangle\langle f_{k'}|_B\right]$$

$$= \sum_{k,k'=1}^{r} s_k s_{k'} \overline{|e_k\rangle}\overline{\langle e_{k'}|}_A\, \underbrace{\mathrm{Tr}\left[|f_k\rangle\langle f_{k'}|\right]}_{=\delta_{k,k'}}$$

$$= \sum_{k=1}^{r} s_k^2 \overline{|e_k\rangle}\overline{\langle e_k|}_A \rightarrow \text{Diagonal!}$$

Also, $\mathrm{Tr}_A\left[|\psi\rangle\langle\psi|_{AB}\right] = \sum_{k=1}^{r} s_k^2 |f_k\rangle\langle f_k|$

$\Rightarrow \mathrm{Tr}_A[\psi_{AB}]$ and $\mathrm{Tr}_B[\psi_{AB}]$ have the same (non-zero) eigenvalues!

(d) Suppose $r=d$, $s_k = \frac{1}{\sqrt{d}}\ \forall\, k\in\{1,2,\dots,d\}$. $\quad |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$

We call $|\psi\rangle$ **maximally entangled** (Note that the marginals are maximally mixed.).

(e) Observe: $\frac{1}{\sqrt{d}} \sum_{k=1}^{d} |e_k\rangle \otimes |\bar{e}_k\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \left(\sum_{j=0}^{d-1} |j\rangle\langle j| \right) |e_k\rangle \otimes \left(\sum_{l=0}^{d-1} |l\rangle\langle l| \right) |\bar{e}_k\rangle$

$$= \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \sum_{j,l=0}^{d-1} |j\rangle \otimes |l\rangle \langle j|e_k\rangle \underbrace{\langle l|\bar{e}_k\rangle}_{= \overline{\langle \bar{e}_k|l\rangle} = \langle e_k|l\rangle}$$

$$= \frac{1}{\sqrt{d}} \sum_{k=1}^{d} \sum_{j,l=0}^{d-1} |j\rangle \otimes |l\rangle \langle j|e_k\rangle\langle e_k|l\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{j,l=0}^{d-1} |j\rangle \otimes |l\rangle \langle j| \underbrace{\left(\sum_{k=1}^{d} |e_k\rangle\langle e_k| \right)}_{= \mathbb{1}_d \text{. b/c } \{|e_k\rangle\} \text{ is ONB!}} |l\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{j,l=0}^{d-1} |j,l\rangle \langle j|l\rangle$$

$$= \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j,j\rangle$$

$$= |\Phi_d^+\rangle.$$

(f) Vectorized unitaries are maximally entangled

$$|\psi\rangle_{AB} = (U \otimes \mathbb{1}) |\Phi_d^+\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \underbrace{U|k\rangle}_{\equiv |e_k\rangle} \otimes \underbrace{|k\rangle}_{\equiv |f_k\rangle}$$

U is unitary

Also, $\mathrm{Tr}_B\left[|\psi\rangle\langle\psi|_{AB} \right] = \mathrm{Tr}_B\left[(U \otimes \mathbb{1}_B) \Phi^+_{AB} (U^\dagger \otimes \mathbb{1}_B) \right]$

$$= U\,\underbrace{\mathrm{Tr}_B\left[\Phi^+_{AB}\right]}_{=\frac{\mathbb{1}_A}{d}}\,U^\dagger$$

$$= \frac{1}{d} U U^\dagger$$

$$= \frac{1}{d}\mathbb{1}_A$$

$$\mathrm{Tr}_B\left[|\Phi^+\rangle\langle\Phi^+|_{AB}\right]$$

$$= \mathrm{Tr}_B\left[\frac{1}{d}\sum_{k,k'=0}^{d-1} |k\rangle\langle k'|_A \otimes |k\rangle\langle k'|_B\right]$$

$$= \frac{1}{d}\sum_{k,k'=0}^{d-1} |k\rangle\langle k'|_A\, \underbrace{\mathrm{Tr}\left[|k\rangle\langle k'|_B\right]}_{\langle k|k'\rangle = \delta_{k,k'}}$$

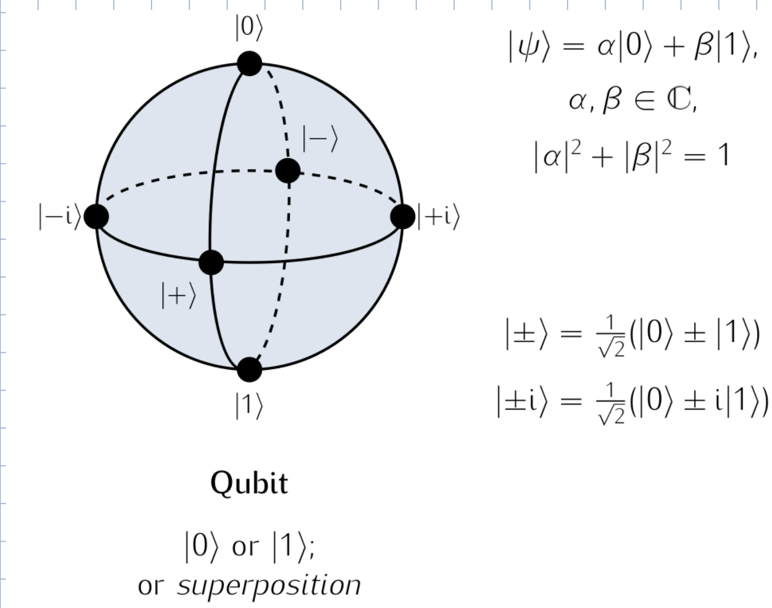
$$= \frac{1}{d}\sum_{k=0}^{d-1} |k\rangle\langle k| = \frac{1}{d}\mathbb{1} \checkmark$$

(Similar for $\mathrm{Tr}_A\left[|\psi\rangle\langle\psi|_{AB}\right]$).

(g) Mixed-state entanglement is much harder to characterize! (More on this later...).

③ Purification of mixed states

(a) Recall: For qubits, pure states are on the surface of the Bloch sphere; mixed states are inside.



General mixed states have the form $\rho = \sum_{k=1}^{N} p_k |\psi_k\rangle\langle\psi_k|$. $\mathrm{Tr}[\rho] = \sum_k p_k = 1$ ✓

probabilities.

(b) How do mixed states arise in nature?

1. Lack of knowledge of state preparation.

Source $\to |\psi_1\rangle$ w/ prob. p_1

Source $\to |\psi_2\rangle$ w/ prob. p_2

$\vdots$

Source $\to |\psi_N\rangle$ w/ prob. p_N.

(*) Now someone uses the source to prepare a state, but they don't tell you which one it is. How do you describe your knowledge of the system?

(They don't mention the label.)

$$\rho_{XS} = \sum_{k=1}^{N} p_k |k\rangle\langle k|_X \otimes |\psi_k\rangle\langle\psi_k|_S$$

(This is a classical-quantum state)

classical variable/register. ("which state is it"?)

$$\begin{pmatrix} p_1|\psi_1\rangle\langle\psi_1| & \\ & p_2|\psi_2\rangle\langle\psi_2| \\ & & \ddots \end{pmatrix}$$

(*) To figure out what state the system is prepared in, we measure the "classical register" X. (w.r.t. POVM $\{|k\rangle\langle k|\}_{k=1}^N$)

$$\mathrm{Tr}\left[\left(|k\rangle\langle k|_X \otimes \mathbb{1}_S\right)\rho_{XS}\right] = \mathrm{Tr}\left[\left(|k\rangle\langle k|_X \otimes \mathbb{1}_S\right)\sum_{k'=1}^{N} p_{k'} |k'\rangle\langle k'|_X \otimes |\psi_{k'}\rangle\langle\psi_{k'}|_S\right]$$

$$= \sum_{k'=1}^{N} \mathrm{Tr}\left[p_{k'}\, |k\rangle\underbrace{\langle k|k'\rangle}_{\delta_{k,k'}}\langle k'|_X \otimes |\psi_{k'}\rangle\langle\psi_{k'}|_S\right]$$

$$= p_k \checkmark \quad \text{(as expected!)}$$

State conditioned on outcome k (i.e., you know what the label is).

$$\mathrm{Tr}_X\left[\frac{\left(|k\rangle\langle k|_X\otimes\mathbb{1}_S\right)\rho_{XS}\left(|k\rangle\langle k|_X\otimes\mathbb{1}_S\right)}{p_k}\right]$$

$$=\frac{1}{p_k}\sum_{k'=1}^{N}\mathrm{Tr}_X\left[\left(|k\rangle\langle k|_X\otimes\mathbb{1}_S\right)\left(p_{k'}|k'\rangle\langle k'|_X\otimes|\psi_{k'}\rangle\langle\psi_{k'}|_S\right)\left(|k\rangle\langle k|_X\otimes\mathbb{1}_S\right)\right]$$

$$=\frac{1}{p_k}\sum_{k'=1}^{N}p_{k'}\underbrace{\mathrm{Tr}_X\left[|k\rangle\langle k|k'\rangle\langle k'|k\rangle\langle k|_X\otimes|\psi_{k'}\rangle\langle\psi_{k'}|_S\right]}_{=\delta_{k',k}\,|\psi_{k'}\rangle\langle\psi_{k'}|}$$

$$=\frac{1}{p_k}p_k|\psi_k\rangle\langle\psi_k|$$

$$=|\psi_k\rangle\langle\psi_k|\ \checkmark\quad\text{(as expected!)}$$

⊛ But if you don't know the outcome/label? $\longrightarrow$ Trace out the register X!

$$\mathrm{Tr}_X(\rho_{XS})=\mathrm{Tr}_X\left[\sum_{k=1}^{N}p_k|k\rangle\langle k|_X\otimes|\psi_k\rangle\langle\psi_k|_S\right]$$

$$=\sum_{k=1}^{N}p_k\underbrace{\mathrm{Tr}_X\left[|k\rangle\langle k|_X\otimes|\psi_k\rangle\langle\psi_k|_S\right]}_{=\underbrace{\mathrm{Tr}\left[|k\rangle\langle k|\right]}_{=1}|\psi_k\rangle\langle\psi_k|}$$

$$=\sum_{k=1}^{N}p_k|\psi_k\rangle\langle\psi_k|.$$

2. Measurement on one part of a pure state.

$$|\psi\rangle_{AB}=\sum_{k=1}^{r}\sqrt{\lambda_k}|e_k\rangle_A\otimes|f_k\rangle_B\longrightarrow\text{Measure } A \text{ wrt POVM } \{M_A^x\}_{x\in X}$$

Conditioned on outcome x, state on B is $\frac{1}{p_x}\mathrm{Tr}_A\left[\left(\sqrt{M_A^x}\otimes\mathbb{1}_B\right)|\psi\rangle\langle\psi|_{AB}\left(\sqrt{M_A^x}\otimes\mathbb{1}_B\right)\right]$

$$\rho_x = \mathrm{Tr}\left[\left(M_A^x \otimes \mathbb{1}_B\right)|\gamma\rangle\langle\gamma|_{AB}\right] = \mathrm{Tr}\Big[M_A^x \underbrace{\mathrm{Tr}_B\left[|\gamma\rangle\langle\gamma|_{AB}\right]}\Big] = \sum_{k=1}^{r} \lambda_k \mathrm{Tr}\left[M_A^x |e_k\rangle\langle e_k|_A\right].$$

$$= \mathrm{Tr}_B\left[\sum_{k,k'=1}^{r} \sqrt{\lambda_k\lambda_{k'}}\, |e_k\rangle\langle e_{k'}|_A \otimes |f_k\rangle\langle f_{k'}|_B\right]$$

$$= \sum_{k,k'=1}^{r} \sqrt{\lambda_k\lambda_{k'}}\, |e_k\rangle\langle e_{k'}|_A \underbrace{\mathrm{Tr}\left[|f_k\rangle\langle f_{k'}|_B\right]}_{=\delta_{k,k'}}$$

$$= \sum_{k=1}^{r} \lambda_k |e_k\rangle\langle e_k|_A$$

$$\mathrm{Tr}_A\left[\left(\sqrt{M_A^x} \otimes \mathbb{1}_B\right)|\gamma\rangle\langle\gamma|_{AB}\left(\sqrt{M_A^x} \otimes \mathbb{1}_B\right)\right] = \mathrm{Tr}_A\left[\sum_{k,k'=1}^{r} \sqrt{M_A^x}\, |e_k\rangle\langle e_{k'}|_A \sqrt{M_A^x} \otimes |f_k\rangle\langle f_{k'}|_B\right]$$

$$= \sum_{k,k'=1}^{r} \mathrm{Tr}\left[\sqrt{M_A^x}\, |e_k\rangle\langle e_{k'}|_A \sqrt{M_A^x}\right] |f_k\rangle\langle f_{k'}|_B$$

$$= \sum_{k,k'=1}^{r} \mathrm{Tr}\left[M_A^x |e_k\rangle\langle e_{k'}|_A\right] |f_k\rangle\langle f_{k'}|_B.$$ $\rightarrow$ Not a pure state in general.

3. Entanglement with an (unknown) environment.

Consider a mixed state $\rho_S = \sum_{k=1}^{r} \underbrace{\lambda_k}_{\text{eigenvalues}} \underbrace{|\gamma_k\rangle}_{\text{eigenvectors (orthonormal)}}\langle\gamma_k|$ (spectral decomposition).

⊛ Note : $\mathrm{Tr}(\rho_S) = 1 \Rightarrow \mathrm{Tr}\left[\sum_{k=1}^{r} \lambda_k \underbrace{\mathrm{Tr}\left[|\gamma_k\rangle\langle\gamma_k|\right]}_{=1}\right] = \sum_{k=1}^{r} \lambda_k = 1$ (eigenvalues sum to one).

⊛ Note : $\rho_S \geq 0 \Rightarrow \lambda_k \geq 0 \ \forall\ k. \Rightarrow 0 \leq \lambda_k \leq 1 \ \forall\ k.$

⊛ There exists a pure state $\underset{\substack{\uparrow \\ \text{environment}}}{|\psi\rangle_{ES}}$ such that $\mathrm{Tr}_E\left[|\psi\rangle\langle\psi|_{ES}\right] = \rho_S$.

<u>Proof</u>: (by explicit construction) Consider $\sqrt{\rho_S} = \sum_{k=1}^{r} \sqrt{\lambda_k}\, |\psi_k\rangle\langle\psi_k|$

Then $$\underset{\substack{\uparrow \\ \dim(\mathcal{H}_E) \geq r}}{|\psi\rangle_{ES}} = \left(\mathbb{1}_E \otimes \sqrt{\rho_S}\right) \underset{\substack{\uparrow \\ |\Gamma\rangle = \sum_{k=0}^{d-1} |k,k\rangle}}{|\Gamma\rangle} = \left(\mathbb{1}_E \otimes \sum_{k=1}^{r} \sqrt{\lambda_k}\, |\psi_k\rangle\langle\psi_k|_S\right)|\Gamma\rangle$$

$$= \sum_{k=1}^{r} \sqrt{\lambda_k}\, \underbrace{\left(\mathbb{1}_E \otimes |\psi_k\rangle\langle\psi_k|_S\right)|\Gamma\rangle_{ES}}_{|\bar{\psi}_k\rangle_E \otimes |\psi_k\rangle_S} = \underline{\sum_{k=1}^{r} \sqrt{\lambda_k}\, |\bar{\psi}_k\rangle_E \otimes |\psi_k\rangle_S}.$$

- This is a state vector: $$\langle\psi|\psi\rangle = \left(\sum_{k=1}^{r} \sqrt{\lambda_k}\, \langle\bar{\psi}_k|_E \otimes \langle\psi_k|_S\right)\left(\sum_{k'=1}^{r} \sqrt{\lambda_{k'}}\, |\bar{\psi}_{k'}\rangle_E \otimes |\psi_{k'}\rangle_S\right)$$

$$= \sum_{k,k'=1}^{r} \sqrt{\lambda_k \lambda_{k'}}\, \underbrace{\langle\bar{\psi}_k|\bar{\psi}_{k'}\rangle}_{\delta_{k,k'}} \underbrace{\langle\psi_k|\psi_{k'}\rangle}_{\delta_{k,k'}}$$

$$= \sum_{k=1}^{r} \lambda_k$$

$$= 1 \quad \checkmark$$

- $$\mathrm{Tr}_E\left[|\psi\rangle\langle\psi|_{ES}\right] = \mathrm{Tr}_E\left[\left(\mathbb{1}_E \otimes \sqrt{\rho_S}\right)|\Gamma\rangle\langle\Gamma|_{ES}\left(\mathbb{1}_E \otimes \sqrt{\rho_S}\right)\right]$$

$$= \sqrt{\rho_S}\, \underbrace{\mathrm{Tr}_E\left[|\Gamma\rangle\langle\Gamma|_{ES}\right]}\, \sqrt{\rho_S}$$

$$= \sum_{k,k'=0}^{d-1} \mathrm{Tr}_E\left[|k\rangle\langle k'|_E \otimes |k\rangle\langle k'|_S \right]$$

$$= \sum_{k,k'=0}^{d-1} \delta_{k,k'} \, |k\rangle\langle k'|_S$$

$$= \sum_{k=0}^{d-1} |k\rangle\langle k|_S = \mathbb{1}_S .$$

$\sqrt{\rho}\,\mathbb{1}\sqrt{\rho} = \rho$

$= \rho_S$ ✓