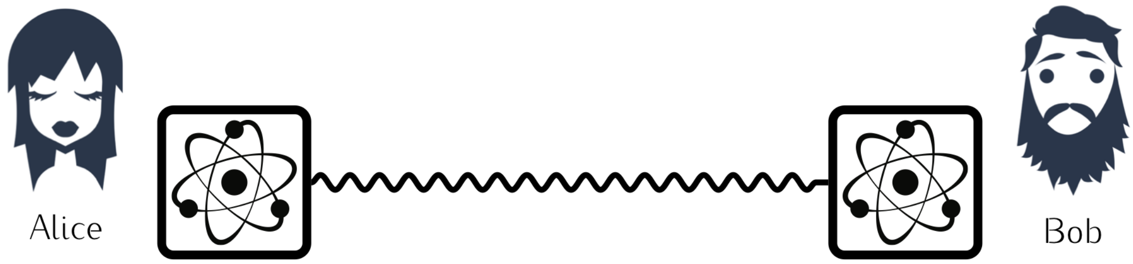
① Recap: Entanglement

- An **entangled state** is NOT a separable state.



$$\rho_{AB} \neq \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \otimes \tau_B^x$$

Entangled state

Correlations between Alice and Bob are non-local.
State of the individual systems not sufficient to describe the pair.

- **Example:** 1. Bell States $|\Phi_{z,x}\rangle = (Z^z X^x \otimes \mathbb{1})|\Phi^+\rangle$, $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$

$$|\Phi_{0,0}\rangle = |\Phi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{array}{l} 00 \\ 01 \\ 10 \\ 11 \end{array}$$

⊛ These states are <u>maximally entangled</u> (more on that later...)

$$|\Phi_{0,1}\rangle = (X \otimes \mathbb{1})|\Phi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$|0,1\rangle + |1,0\rangle$

$$|\Phi_{1,0}\rangle = (Z \otimes \mathbb{1})|\Phi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$|0,0\rangle - |1,1\rangle$

$$|\Phi_{1,1}\rangle = (ZX \otimes \mathbb{1})|\Phi^+\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Phi_{AB}^{z,x} = |\Phi_{z,x}\rangle\langle\Phi_{z,x}|_{AB}$$

2. Convex mixtures of Bell states might be entangled:

$$\rho_{AB} = \sum_{z,x=0}^{1} p_{z,x} \Phi_{AB}^{z,x}.$$

$p_{z,x}$ ↳ probabilities

→ Isotropic states: $\rho_{AB}^{(\lambda)} = (1-\lambda)\Phi_{0,0} + \frac{\lambda}{3}(\Phi_{0,1} + \Phi_{1,0} + \Phi_{1,1})$

$p_{0,0} = 1-\lambda$, $p_{0,1} = p_{1,1} = p_{1,1} = \frac{\lambda}{3}$, $\lambda \in [0,1]$

$(U \otimes \bar{U})\rho_{AB}^{(\lambda)}(U \otimes \bar{U})^\dagger = \rho_{AB}^{(\lambda)}$
$\forall$ unitaries U.

For $\lambda = 3/4$, $\rho_{AB}^{(3/4)} = \frac{1}{4}\underbrace{(\Phi_{0,0} + \Phi_{0,1} + \Phi_{1,0} + \Phi_{1,1})}_{\mathbb{1}_A \otimes \mathbb{1}_B.} = \underbrace{\frac{1}{4}\mathbb{1}_A \otimes \mathbb{1}_B}$ ↳ $\frac{\mathbb{1}_A}{2} \otimes \frac{\mathbb{1}_B}{2}$

→ Werner states: $\omega_{AB}^{(\lambda)} = (1-\lambda)\Phi_{1,1} + \frac{\lambda}{3}(\Phi_{0,0} + \Phi_{0,1} + \Phi_{1,0})$, $\lambda \in [0,1]$.

$(U \otimes U)\omega_{AB}^{(\lambda)}(U \otimes U)^\dagger = \omega_{AB}^{(\lambda)}$
$\forall$ unitaries U.

→ $\rho_{AB} = (a+b)\Phi_{0,0} + (a-b)\Phi_{1,0} + c\Phi_{0,1} + c\Phi_{1,1}$

Entangled if $a+b > \frac{1}{2}$. → proof later.

<u>Note</u>: $a+b = \langle \Phi^+ | \rho_{AB} | \Phi^+ \rangle$ → <u>fidelity</u>

3. Multi-party entangled states:

<u>GHZ State</u>: $|GHZ_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$.

$n=2 \Rightarrow$ Bell State

$|GHZ_3\rangle = \frac{1}{\sqrt{2}}(|0,0,0\rangle + |1,1,1\rangle)$.

GOING BEYOND BELL'S THEOREM (1989) (arXiv:0712.0921)

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- Why is entanglement special?

arXiv > quant-ph > arXiv:quant-ph/0702225

Quantum Physics

[Submitted on 26 Feb 2007 (v1), last revised 20 Apr 2007 (this version, v2)]

Quantum entanglement

Ryszard Horodecki, Pawel Horodecki, Michal Horodecki, Karol Horodecki

arXiv > quant-ph > arXiv:1806.06107

Quantum Physics

[Submitted on 15 Jun 2018 (v1), last revised 20 Apr 2019 (this version, v3)]

Quantum Resource Theories

Eric Chitambar, Gilad Gour

→ QKD (Quantum key distribution)

→ Teleportation

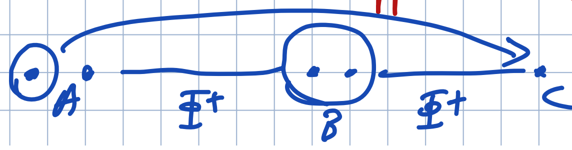
→ Superdense coding

→ Error correction

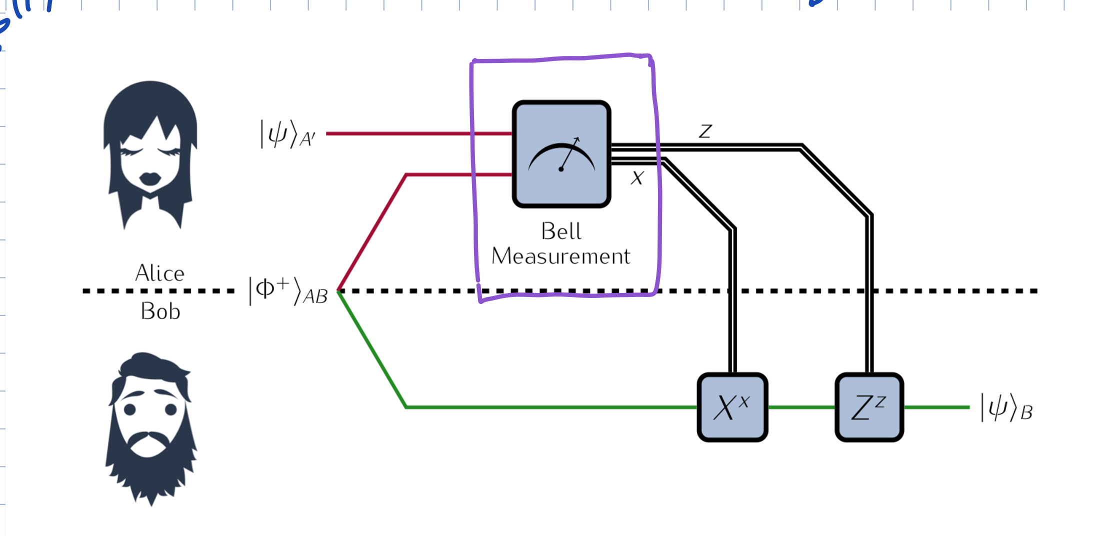
→ Measurement-based quantum computing

→ Quantum sensing.

→ Distributed applications (i.e, in a network).



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$



② Measurements in Quantum mechanics

- Recall: $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow$ Probability of 0 : $|\alpha|^2$

  Probability of 1 : $|\beta|^2$

- Observe: $\mathrm{Tr}[\,|0\rangle\langle 0|\,|\psi\rangle\langle\psi|\,]$ → (Assignment: $\mathrm{Tr}[M|\psi\rangle\langle\psi|] = \langle\psi|M|\psi\rangle$

  Hint: Use definition of trace: $\mathrm{Tr}[M|\psi\rangle\langle\phi|] = \sum_{k=0}^{d-1}\langle k|M|\psi\rangle\langle\phi|k\rangle$)

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \underbrace{\langle 0|\psi\rangle}_{\alpha}\underbrace{\langle\psi|0\rangle}_{\bar{\alpha}}$$

$$= |\alpha|^2$$

$$\mathrm{Tr}[N] = \sum_{k=0}^{d-1} \underbrace{\langle k|N|k\rangle}$$

↳ Diagonal elements.

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Also: $\mathrm{Tr}[\,|1\rangle\langle 1|\,|\psi\rangle\langle\psi|\,] = \underbrace{\langle 1|\psi\rangle}_{\beta}\underbrace{\langle\psi|1\rangle}_{\bar{\beta}} = \beta\cdot\bar{\beta} = |\beta|^2$

- The measurement outcomes are associated with operators $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$

such that $\Pr[0] = \mathrm{Tr}[|0\rangle\langle 0|\psi\rangle\langle\psi|]$, $\Pr[1] = \mathrm{Tr}[|1\rangle\langle 1|\psi\rangle\langle\psi|]$

This is true for density matrices as well!

So $\Pr[0] = \mathrm{Tr}[|0\rangle\langle 0|\rho]$, $\Pr[1] = \mathrm{Tr}[|1\rangle\langle 1|\rho]$

Then $\Pr[0] + \Pr[1] = \mathrm{Tr}[|0\rangle\langle 0|\rho] + \mathrm{Tr}[|1\rangle\langle 1|\rho] = \mathrm{Tr}[|0\rangle\langle 0|\rho + |1\rangle\langle 1|\rho]$

$$= \mathrm{Tr}[\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{=\mathbb{1}}\rho] = \mathrm{Tr}[\rho] = 1 \checkmark$$

Measurement outcome labels.

- POVM (positive operator-valued measure): $\{M_x\}_{x\in\mathcal{X}}$
  - $M_x \geq 0$
  - $\sum_{x\in\mathcal{X}} M_x = \mathbb{1}$
  - $\Pr[x] = \mathrm{Tr}[M_x\rho]$ $\rightarrow$ Born Rule

Check: $\sum_x \Pr[x] = \sum_x \mathrm{Tr}[M_x\rho] = \mathrm{Tr}[(\underbrace{\sum_x M_x}_{\mathbb{1}})\rho] = \mathrm{Tr}[\rho] = 1$

(a) Examples:

1. $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ $\longrightarrow$ "computational basis measurement" "Pauli-Z meas."

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z$

2. $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ $\rightarrow$ "Pauli-X meas." $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

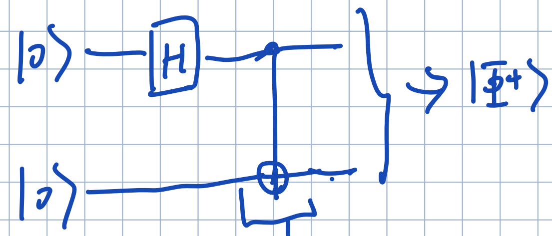
3. $\{|+i\rangle\langle +i|, |-i\rangle\langle -i|\}$ $\rightarrow$ "Pauli-Y meas." $Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\{|e_k\rangle\}_{k=0}^{d-1}$ ONB

4. Every ONB on $\mathbb{C}^d$ defines a measurement.

$\Rightarrow \sum_k \underbrace{|e_k\rangle\langle e_k|}_{M_k} = \mathbb{1}$

5. Bell measurement: $\{\Phi^{zx} : z, x \in \{0,1\}\}$



CNOT gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

(Hadamard gate).

$H|0\rangle = |+\rangle$

$H|1\rangle = |-\rangle$

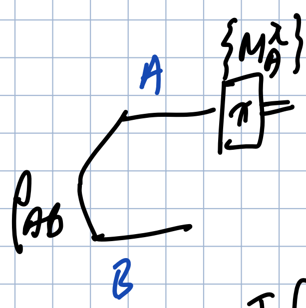
$CNOT|0,0\rangle = |0,0\rangle$

$CNOT|0,1\rangle = |0,1\rangle$

$CNOT|1,0\rangle = |1,1\rangle$

$CNOT|1,1\rangle = |1,0\rangle$

(b) "Partial" measurement: measuring only one part of a state.



$$\Pr[x] \underset{\text{By definition}}{=} \mathrm{Tr}\left[\left(M_A^x \otimes \mathbb{1}_B\right)\rho_{AB}\right] = \sum_{k=0}^{d_A-1}\sum_{l=0}^{d_B-1} \underbrace{\langle k, l|_{AB}}_{= \langle k|_A \otimes \langle l|_B \;=\; \langle k|_A (\mathbb{1}_A \otimes \langle l|_B)} \left(M_A^x \otimes \mathbb{1}_B\right)\rho_{AB}|k, l\rangle_{AB}$$

$N \otimes M$
$= (N \otimes \mathbb{1})(\mathbb{1} \otimes M)$
$= (\mathbb{1} \otimes M)(N \otimes \mathbb{1})$

$$= \mathrm{Tr}\left[M_A^x \rho_A\right]$$

$$= \sum_{k=0}^{d_A-1}\sum_{l=0}^{d_B-1} \langle k|_A \underbrace{(\mathbb{1}_A \otimes \langle l|_B)(M_A^x \otimes \mathbb{1}_B)}_{= M_A^x(\mathbb{1}_A \otimes \langle l|_B)}\rho_{AB}(\mathbb{1}_A \otimes |l\rangle_B)|k\rangle_A$$

$$= \sum_{k=0}^{d_A-1}\sum_{l=0}^{d_B-1} \langle k|_A M_A^x (\mathbb{1}_A \otimes \langle l|_B)\rho_{AB}(\mathbb{1}_A \otimes |l\rangle_B)|k\rangle_A$$

$$= \sum_{k=0}^{d_A-1} \langle k|_A M_A^x \underbrace{\sum_{l=0}^{d_B-1}\left((\mathbb{1}_A \otimes \langle l|_B)\rho_{AB}(\mathbb{1}_A \otimes |l\rangle_B)\right)}_{= \rho_A \;\to\; \text{State from Alice's perspective}}|k\rangle_A$$

$$= \sum_{k=0}^{d_A-1} \langle k|_A M_A^x \rho_A |k\rangle$$

$$= \mathrm{Tr}\left[M_A^x \rho_A\right].$$ $\longrightarrow$ Probability only involves state from Alice's perspective!

(*) $\rho_A \equiv \mathrm{Tr}_B[\rho_{AB}]$ is called the partial trace of ρ_{AB} over B.

$\mathrm{Tr}_A[\rho_{AB}] \equiv \rho_B$

- $\mathrm{Tr}_A[M_A \otimes N_B] = \mathrm{Tr}[M_A]\, N_B\,, \quad \mathrm{Tr}_B[M_A \otimes N_B] = M_A\, \mathrm{Tr}[N_B]$

<u>Proof</u>: $$\mathrm{Tr}_A[M_A \otimes N_B] = \sum_{k=0}^{d_A-1} \left(\langle k|_A \otimes \mathbb{1}_B\right)\left(M_A \otimes N_B\right)\left(|k\rangle_A \otimes \mathbb{1}_B\right)$$

$$= \sum_{k=0}^{d_A-1} \langle k|M_A|k\rangle \cdot N_B$$

$$= \mathrm{Tr}[M_A]\, N_B$$

$$\mathrm{Tr}_B[M_A \otimes N_B] = \sum_{k=0}^{d_B-1} \left(\mathbb{1}_A \otimes \langle k|_B\right)\left(M_A \otimes N_B\right)\left(\mathbb{1}_A \otimes |k\rangle_B\right)$$

$$= \sum_{k=0}^{d_B-1} M_A \cdot \langle k|N_B|k\rangle$$

$$= M_A \left(\sum_{k=0}^{d_A-1} \langle k|N_B|k\rangle\right)$$

$$= \mathrm{Tr}[N_B]\, M_A\,. \quad \blacksquare$$

- <u>Recall full trace</u>: $\mathrm{Tr}[N_{AB}] = \sum_{k=0}^{d_A-1} \sum_{l=0}^{d_B-1} \langle k,l|_{AB} N_{AB} |k,l\rangle_{AB}$.

- $\mathrm{Tr}_A[N_{AB}] = \sum_{k=0}^{d_A-1} (\langle k|_A \otimes \mathbb{1}_B) N_{AB} (|k\rangle_A \otimes \mathbb{1}_B)$

- <u>Example</u>: Partial trace of Bell States.

$$\Phi^+_{AB} = |\Phi^+\rangle\langle\Phi^+|_{AB}, \quad |\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle).$$

$$\mathrm{Tr}_A[\Phi^+_{AB}] = (\langle 0|_A \otimes \mathbb{1}_B)|\Phi^+\rangle\langle\Phi^+|_{AB}(|0\rangle_A \otimes \mathbb{1}_B)$$
$$+ (\langle 1|_A \otimes \mathbb{1}_B)|\Phi^+\rangle\langle\Phi^+|_{AB}(|1\rangle_A \otimes \mathbb{1}_B)$$

$$\hookrightarrow = \frac{1}{\sqrt{2}}\Big((\langle 0|_A \otimes \mathbb{1}_B)|0,0\rangle_{AB} + (\langle 0|_A \otimes \mathbb{1}_B)|1,1\rangle_{AB}\Big) \to = \langle 0|0\rangle |0\rangle_B$$
$$= \frac{1}{\sqrt{2}}\Big(\cancel{\langle 0|0\rangle}|0\rangle_B + \cancel{\langle 0|1\rangle}|1\rangle_B\Big)$$
$$= \frac{1}{\sqrt{2}}|0\rangle_B$$

<u>Similarly</u>: $(\langle 1|_A \otimes \mathbb{1}_B)|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}|1\rangle_B$.

$$= \frac{1}{2}|0\rangle\langle 0|_B + \frac{1}{2}|1\rangle\langle 1|_B = \frac{\mathbb{1}_B}{2} \to \text{maximally mixed!}$$

(*) Also, $\mathrm{Tr}_B[\Phi^+_{AB}] = \dfrac{\mathbb{1}_A}{2}$

(*) If only Alice measures her qubit, then her outcomes are completely random!

$$\mathrm{Tr}[(|0\rangle\langle 0|_A \otimes \mathbb{1}_B)\Phi^+_{AB}] = \mathrm{Tr}[|0\rangle\langle 0|_A \underline{\mathrm{Tr}_B[\Phi^+_{AB}]}] = \mathrm{Tr}[|0\rangle\langle 0|_A \frac{\mathbb{1}_A}{2}] = \frac{1}{2}$$

⊛ But measuring jointly, the outcomes are correlated!

$$\Pr[0,0] = \mathrm{Tr}\left[|0,0\rangle\langle 0,0|\,\Phi^+_{AB}\right] = \underbrace{\langle 0,0|\Phi^+\rangle}_{=\frac{1}{\sqrt{2}}}\underbrace{\langle\Phi^+|0,0\rangle}_{=\frac{1}{\sqrt{2}}} = \frac{1}{2}$$

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\underbrace{\left(|0,0\rangle + |1,1\rangle\right)}$$

$$\Pr[0,1] = \mathrm{Tr}\left[|0,1\rangle\langle 0,1|\,\Phi^+_{AB}\right] = \langle 0,1|\Phi^+\rangle\langle\Phi^+|0,1\rangle = 0$$

$$\Pr[1,0] = \mathrm{Tr}\left[|1,0\rangle\langle 1,0|\,\Phi^+_{AB}\right] = 0$$

$$\Pr[1,1] = \mathrm{Tr}\left[|1,1\rangle\langle 1,1|\,\Phi^+_{AB}\right] = \frac{1}{2}$$

⊛ This is true in any basis!