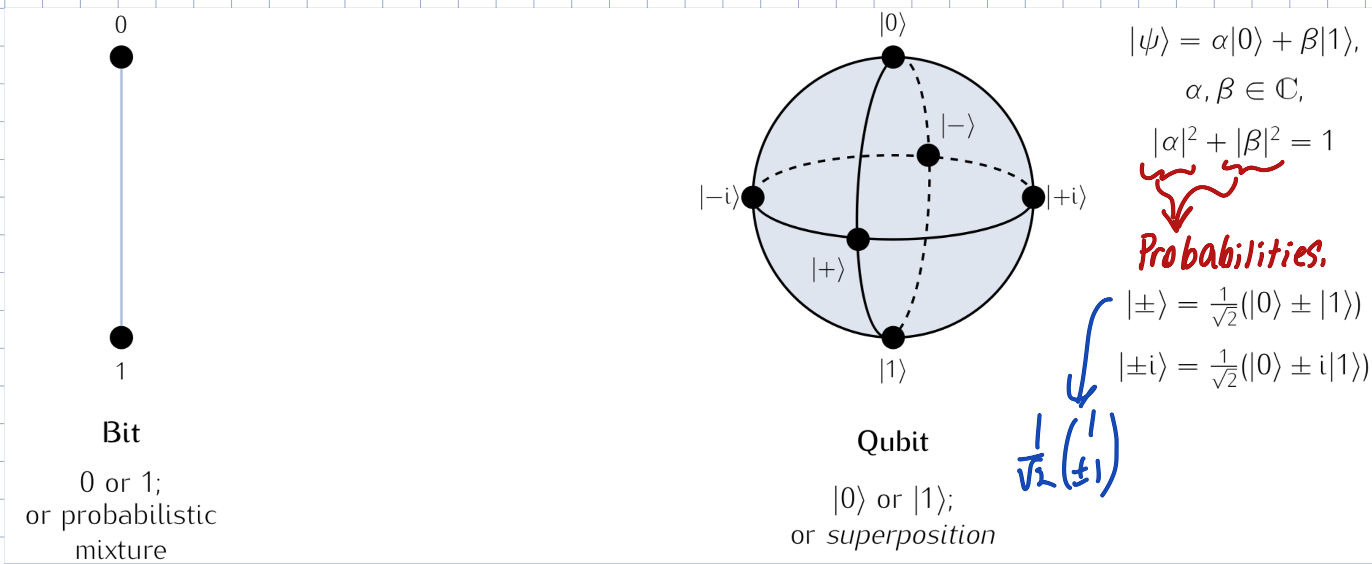
① Recap: Math Background

(a) **Vectors:** $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\rightarrow |\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = c_0|0\rangle + c_1|1\rangle$

Two-dimensional "Hilbert Space" $\mathbb{C}^2$ (describes the state of a qubit)



Probabilities.

$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix}$

For dimension $d \in \{2, 3, \ldots\}$: $|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\ldots$, $|d-1\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$

$\underline{d=3}$ $|0\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|1\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$|\psi\rangle = \sum_{k=0}^{d-1} c_k |k\rangle = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{d-1} \end{pmatrix}$

$\in \mathbb{C}$

$\rightarrow$ The norm is $\| |\psi\rangle \| = \left(\sum_{k=0}^{d-1} |c_k|^2 \right)^{1/2}$

$\rightarrow c_k = \langle k | \psi \rangle$

$\langle 1 | \psi \rangle = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = c_1$ ✓

$(0\ 1\ 0)$

⊛ For a state vector: $\sum_{k=0}^{d-1} |c_k|^2 = 1$.

Probability of finding the qudit in state $|k\rangle$.

⊛ $\{|k\rangle\}_{k=0}^{d-1}$ is an orthonormal basis

$\langle k | k' \rangle = \delta_{k,k'} = \begin{cases} 1 & \text{if } k = k' \\ 0 & \text{if } k \neq k' \end{cases}$

⊛ For $|\psi\rangle = \sum_{k=0}^{d-1} c_k |k\rangle = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{pmatrix}$ ("ket") (column vector)

$\rightarrow \langle\psi| = \sum_{k=0}^{d-1} \bar{c}_k \langle k| = \begin{pmatrix} \bar{c}_0 & \bar{c}_1 & \cdots & \bar{c}_{d-1} \end{pmatrix}$ ("bra") (row vector).

⊛ <u>Inner Product</u>: $\langle\phi|\psi\rangle$; $|\phi\rangle = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix}$, $|\psi\rangle = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{pmatrix}$

$$\Rightarrow \langle\phi|\psi\rangle = \begin{pmatrix} \bar{a}_0 & \bar{a}_1 & \cdots & \bar{a}_{d-1} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{pmatrix} = \bar{a}_0 c_0 + \bar{a}_1 c_1 + \dots + \bar{a}_{d-1} c_{d-1} = \sum_{k=0}^{d-1} \bar{a}_k c_k$$

<u>Note</u>: $\langle\psi|\phi\rangle = \overline{\langle\phi|\psi\rangle}$

<u>Proof</u>: $\langle\phi|\psi\rangle = \sum_{k=0}^{d-1} \bar{a}_k c_k \Rightarrow \overline{\langle\phi|\psi\rangle} = \overline{\sum_{k=0}^{d-1} \bar{a}_k c_k} = \sum_{k=0}^{d-1} \overline{\bar{a}_k c_k} = \sum_{k=0}^{d-1} \underbrace{\bar{\bar{a}}_k}_{=a_k} \bar{c}_k = \sum_{k=0}^{d-1} \bar{c}_k a_k$

But also $\langle\psi|\phi\rangle = \begin{pmatrix} \bar{c}_0 & \bar{c}_1 & \cdots & \bar{c}_{d-1} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{d-1} \end{pmatrix} = \sum_{k=0}^{d-1} \bar{c}_k a_k = \overline{\langle\phi|\psi\rangle}$. ∎

(b) <u>Matrices</u> : "unit" matrices: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 0|$, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = |0\rangle\langle 1|$

(Used for describing general quantum states and quantum gates.)

$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |1\rangle\langle 0|$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|$

$$M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} = m_{00}\,|0\rangle\langle 0| + m_{01}\,|0\rangle\langle 1| + m_{10}\,|1\rangle\langle 0| + m_{11}\,|1\rangle\langle 1|$$

$$m_{kl} = \langle k|M|l\rangle$$

e.g.,: $\langle 0|M|0\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} m_{00} & m_{01} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = m_{00}$ ✓

- For a $d \times d$ matrix: $M = \sum_{k,l=0}^{d-1} m_{k,l}\,|k\rangle\langle l|$, $m_{k,l} = \langle k|M|l\rangle$

- Identity matrix: $\mathbb{1} = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \\ & & & & 1 \end{pmatrix} = \sum_{k=0}^{d-1} |k\rangle\langle k| = \sum_{k=0}^{d-1} |e_k\rangle\langle e_k|$

$$\mathbb{1}|\psi\rangle = |\psi\rangle$$

- Conjugate Transpose: $M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix}$

$$\Rightarrow M^\dagger = \begin{pmatrix} \overline{m_{00}} & \overline{m_{10}} \\ \overline{m_{01}} & \overline{m_{11}} \end{pmatrix}$$

- Hermitian matrix: $M^\dagger = M \Rightarrow M = \begin{pmatrix} m_{00} & m_{01} \\ m_{10} & m_{11} \end{pmatrix} = \begin{pmatrix} \overline{m_{00}} & \overline{m_{10}} \\ \overline{m_{01}} & \overline{m_{11}} \end{pmatrix}$

$\Rightarrow m_{00} = \overline{m_{00}} \Rightarrow m_{00}$ is real! $(m_{00} \in \mathbb{R})$

$m_{01} = \overline{m_{10}}$, $m_{11} = \overline{m_{11}} \Rightarrow m_{11} \in \mathbb{R}$

$$\Rightarrow M = \begin{pmatrix} m_{00} & m_{01} \\ \overline{m_{01}} & m_{11} \end{pmatrix}$$

• **Density Matrix:** (Points inside the Bloch Sphere.)

- $\rho^{\dagger} = \rho$
- $\rho \geq 0$ (non-negative eigenvalues).
- $\underbrace{Tr[\rho]}$ $= 1$

↳ "trace" → sum of the diagonal elements.

$$Tr[M] = \sum_{k=0}^{d-1} \langle k|M|k\rangle$$

(c) **Tensor Product:** (For describing multiple qubits, entanglement.)

$$|\psi_1\rangle = \begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}, \quad |\psi_2\rangle = \begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix} \Rightarrow |\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix}\alpha_1\alpha_2\\ \alpha_1\beta_2\\ \beta_1\alpha_2\\ \beta_1\beta_2\end{pmatrix}$$

$$\underset{1\times 2}{\begin{pmatrix}\alpha_1 & \beta_1\end{pmatrix}} \otimes \underset{2\times 1}{\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}} = \begin{pmatrix}\alpha_1\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix} & \beta_1\begin{pmatrix}\alpha_2\\ \beta_2\end{pmatrix}\end{pmatrix} \rightarrow \underset{2\times 2}{\left(\begin{array}{c|c}\alpha_1\alpha_2 & \beta_1\alpha_2\\ \hline \alpha_1\beta_2 & \beta_1\beta_2\end{array}\right)}$$

• For dimension d: $|\psi_1\rangle = \sum_{k=0}^{d-1} a_k|k\rangle$, $|\psi_2\rangle = \sum_{l=0}^{d-1} c_l|l\rangle$

$$\Rightarrow |\psi_1\rangle \otimes |\psi_2\rangle = \left(\sum_{k=0}^{d-1} a_k|k\rangle\right) \otimes \left(\sum_{l=0}^{d-1} c_l|l\rangle\right)$$

$$= \sum_{k,l=0}^{d-1} a_k c_l \underbrace{|k\rangle \otimes |l\rangle}$$ → Also written as $|k,l\rangle$

$$\underbrace{|0\rangle \otimes |0\rangle}_{|0,0\rangle} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$|0,0\rangle\langle 0,0| = \begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

② Entanglement

✱ State vectors of two qubits belongs to the tensor-product space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

(a) – A **product state (vector)** has the form:

(State vector) $|\psi\rangle_{AB} = |\phi_1\rangle_A \otimes |\phi_2\rangle_B$, $|\phi_1\rangle \in \mathbb{C}^{d_1}$, $|\phi_2\rangle \in \mathbb{C}^{d_2}$.

(Density operator). $\rho_{AB} = \sigma_A \otimes \tau_B$, $\sigma_A \in D(\mathbb{C}^{d_1})$, $\tau_B \in D(\mathbb{C}^{d_2})$. set of density matrices.

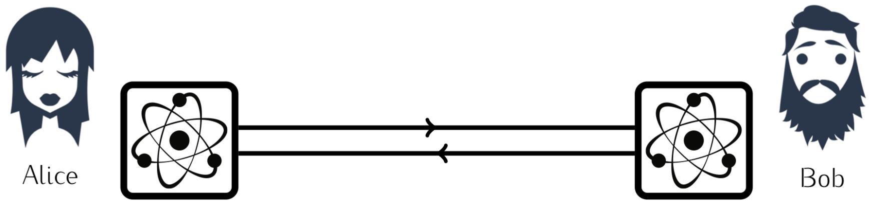
$$\rho_{AB} = \sigma_A \otimes \tau_B$$

Product state

Alice and Bob individually prepare their systems.

– A **separable state** has the form: $\rho_{AB} = \sum_x p(x)\, \sigma_A^x \otimes \tau_B^x$

↳ Also called "classically correlated" ↳ probabilities.



$$\rho_{AB} = \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \otimes \tau_B^x$$

Separable state

Alice and Bob individually prepare their systems via local operations and classical communication.

Example: $\rho_{AB} = \frac{1}{2}\left(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B \right). \to \mathcal{X} = \{0,1\},\ p(0) = p(1) = \frac{1}{2}$

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Aside: This state is **diagonal** in the tensor-product basis

$$\{ |0\rangle_A \otimes |0\rangle_B,\ |0\rangle_A \otimes |1\rangle_B,\ |1\rangle_A \otimes |0\rangle_B,\ |1\rangle_A \otimes |1\rangle_B \}$$

$\rho_{AB} = \frac{1}{2}|0,0\rangle\langle 0,0|_{AB} + \frac{1}{2}|1,1\rangle\langle 1,1|_{AB}$

$$\rho_{AB} = \frac{1}{2} \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} \end{array} \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix}$$

All states of two probabilistic bits can be expressed in this way!

$$\rho_{AB} = \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix} \begin{array}{c} \begin{pmatrix} p_1 & & 0 & \\ & p_2 & & \\ & & p_3 & \\ & 0 & & p_4 \end{pmatrix} \\ \begin{matrix} 00 & 01 & 10 & 11 \end{matrix} \end{array}$$

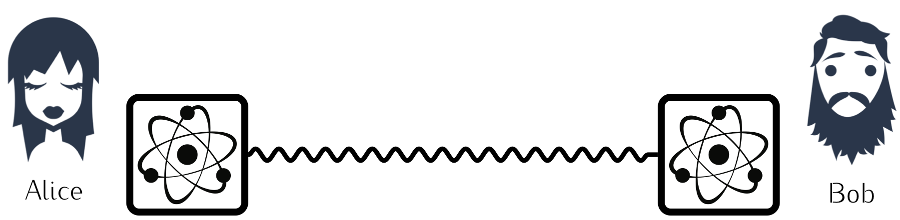
(✱ Even one bit!

$\rho = \begin{pmatrix} p_1 & \\ & p_2 \end{pmatrix} = \frac{1}{2}\left(p_1|0\rangle\langle 0| + p_2|1\rangle\langle 1|\right)$

describes an arbitrary state of one bit!)

✱ Any system whose states are always diagonal in a fixed basis is referred to as a "classical system".

- An **entangled state** is NOT a separable state.



$$\rho_{AB} \neq \sum_{x \in \mathcal{X}} p(x) \sigma_A^x \otimes \tau_B^x$$

Entangled state

Correlations between Alice and Bob are non-local.
State of the individual systems not sufficient to describe the pair.

(b) Examples:

1. The Bell states

$|0,0\rangle \equiv |0\rangle \otimes |0\rangle$
$|1,1\rangle \equiv |1\rangle \otimes |1\rangle$

$$\Phi^{\pm} = |\Phi^{\pm}\rangle\langle\Phi^{\pm}|, \quad |\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle \pm |1,1\rangle\right)$$

$$\Psi^{\pm} = |\Psi^{\pm}\rangle\langle\Psi^{\pm}|, \quad |\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle \pm |1,0\rangle\right).$$

Observe that $\{|\Phi^{\pm}\rangle, |\Psi^{\pm}\rangle\} \longleftrightarrow \{\underbrace{(Z^z X^x \otimes \mathbb{1})|\Phi^{+}\rangle}_{|\Phi_{z,x}\rangle} : z,x \in \{0,1\}\}$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z^0 = \mathbb{1},\ Z^1 = Z, \quad X^0 = \mathbb{1},\ X^1 = X$$

$0,0 \leftrightarrow \Phi^{+}$
$0,1 \leftrightarrow \Psi^{+}$
$1,0 \leftrightarrow \Phi^{-}$
$1,1 \leftrightarrow \Psi^{-}$

In d dimensions: $|\Phi_d^{+}\rangle := \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k,k\rangle$ $\quad \frac{1}{\sqrt{3}}\left(|0,0\rangle + |1,1\rangle + |2,2\rangle\right)$

(See later for the other d-dimensional Bell states.)

$$|\Phi^{+}\rangle = \begin{array}{c} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \end{array} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}, \quad |\Phi^{-}\rangle = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}, \quad |\Psi^{+}\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \quad |\Psi^{-}\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

$|\Phi^{+}\rangle = |\Phi_{0,0}\rangle$
$|\Phi_{0,1}\rangle = |\Psi^{+}\rangle$
$|\Phi_{1,0}\rangle = |\Phi^{-}\rangle$
$|\Phi_{1,1}\rangle = |\Psi^{-}\rangle$

$$(X \otimes \mathbb{1})|\Phi^{+}\rangle = (X \otimes \mathbb{1})\frac{1}{\sqrt{2}}\left(|0,0\rangle + |1,1\rangle\right)$$

$|01\rangle$
$|0,1\rangle \equiv |0\rangle|1\rangle$

$$= \frac{1}{\sqrt{2}}\left(\underbrace{(X \otimes \mathbb{1})|0,0\rangle} + (X \otimes \mathbb{1})|1,1\rangle\right) = \frac{1}{\sqrt{2}}\left(|1,0\rangle + |0,1\rangle\right)$$

$$(X \otimes \mathbb{1})(|0\rangle \otimes |0\rangle) = \underbrace{X|0\rangle}_{|1\rangle} \otimes \underbrace{\mathbb{1}|0\rangle}_{|0\rangle} = |1,0\rangle$$

$$\frac{1}{\sqrt{2}}\left(|0,0\rangle + |1,1\rangle\right)$$

$\circledast$ $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$, $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \alpha_1\alpha_2|0,0\rangle + \alpha_1\beta_2|0,1\rangle + \beta_1\alpha_2|1,0\rangle + \beta_1\beta_2|1,1\rangle$$

There does not exist values of $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $|\psi_1\rangle \otimes |\psi_2\rangle = |\Phi^+\rangle$.

$\circledast$ $\{|\Phi_{z,x}\rangle : z,x \in \{0,1\}\}$ is an ONB for $\mathbb{C}^2 \otimes \mathbb{C}^2$

$|0,0\rangle$
$|0,1\rangle$
$|1,0\rangle$
$|1,1\rangle$

$\langle 0,0|0,1\rangle$
$= \langle 0|0\rangle\langle 0|1\rangle$
$= 0$

$$\underline{\underline{U}}|z,x\rangle = |\Phi_{z,x}\rangle$$

$\hookrightarrow$ U is a unitary matrix : $U^\dagger U = \mathbb{1} = UU^\dagger \Leftrightarrow U^\dagger = U^{-1}$
(change-of-basis matrix).

$$|0,0\rangle\langle 0,1| = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 0 & 1 & 0 & 0 \\ 01 & 0 & 0 & 0 & 0 \\ 10 & 0 & 0 & 0 & 0 \\ 11 & 0 & 0 & 0 & 0 \end{array}$$

$\Rightarrow \{|\Phi_{z,x}\rangle\langle\Phi_{z',x'}| : z,x,z',x' \in \{0,1\}\}$ is an ONB for $L(\mathbb{C}^2 \otimes \mathbb{C}^2)$

$$\Rightarrow \rho_{AB} = \sum_{\substack{z,x \in \{0,1\} \\ z',x' \in \{0,1\}}} \underbrace{C_{z,x}^{z',x'}}_{\in \mathbb{C}} |\Phi_{z,x}\rangle\langle\Phi_{z',x'}|_{AB}$$

$\circledast$ $\sum_{z,x \in \{0,1\}} \Phi_{AB}^{z,x} = \mathbb{1}_{AB} \longrightarrow$ Bell states form a $\underline{\underline{\text{measurement}}}$ (POVM) — see later.

$\hookrightarrow$ Important ingredient in teleportation.

$$\sum_{k=0}^{d} |e_k\rangle\langle e_k| = \mathbb{1}$$