CS 5134 – Introduction to Quantum Computer Science

Assignment 2

Virginia Tech Department of Computer Science, Fall 2025
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Issue date: Tuesday, September 30, 2025

Due date: Tuesday, October 14, 2025

Instructions: Please show all of your work. Your grade will be based not only on getting the correct answer, but having a complete solution. You may work in collaboration with others, but you must write your own solutions, in your own words. Submit in class or via email.

Grading: You will start from a grade of 100%. For every missed step, unclear or unexplained step, and mistake, you will lose two points, with a maximum of 10 points of deduction per Part. Unattempted problems within a Part result in a four-point deduction. Sticking to the deadline is to your benefit, but there is no late penalty. If you would like the assignment to count towards your final grade, then submit it by the final day of classes in December. Bonus problems are optional; they need not be attempted to get a full score on the assignment.

Part 1 *Partial trace.*

1. Calculate the partial trace $\text{Tr}_B[M_{AB}]$ for the following two-qubit linear operators.

   (a) $$M_{AB} = \begin{pmatrix} \frac{1}{3} + \frac{2}{5}\text{i} & 0 & 0 & 0 \\ 0 & -\frac{7}{4} - \frac{1}{9}\text{i} & 0 & 0 \\ 0 & 0 & \frac{5}{2} + \frac{3}{2}\text{i} & 0 \\ 0 & 0 & 0 & -\frac{2}{7} + \frac{1}{8}\text{i} \end{pmatrix}$$

   (b) $$M_{AB} = \begin{pmatrix} \frac{1}{2} - \frac{1}{3}\text{i} & \frac{2}{3} + \frac{3}{4}\text{i} & -\frac{5}{7} & \frac{1}{2}\text{i} \\ -\frac{4}{3} & \frac{7}{5} - \frac{2}{5}\text{i} & \frac{1}{9} + \frac{2}{3}\text{i} & 0 \\ \frac{1}{8} & \frac{4}{5}\text{i} & -\frac{2}{11} & \frac{6}{7} + \frac{3}{8}\text{i} \\ \frac{2}{9}\text{i} & \frac{1}{6} & -\frac{1}{7}\text{i} & \frac{3}{2} - \frac{1}{6}\text{i} \end{pmatrix}$$

   (c) $$M_{AB} = \begin{pmatrix} \frac{3}{5} & \frac{1}{2} - \frac{1}{4}\text{i} & 0 & \frac{1}{3}\text{i} \\ \frac{1}{2} + \frac{1}{4}\text{i} & \frac{2}{3} & \frac{4}{7}\text{i} & 0 \\ 0 & -\frac{4}{7}\text{i} & 1 & -\frac{2}{9} + \frac{1}{2}\text{i} \\ -\frac{1}{3}\text{i} & 0 & -\frac{2}{9} - \frac{1}{2}\text{i} & \frac{4}{9} \end{pmatrix}$$

   (d) $$M_{AB} = \begin{pmatrix} 0 & \frac{5}{7} + \frac{3}{11}\text{i} & \frac{1}{6} - \frac{2}{5}\text{i} & \frac{2}{3} + \frac{1}{4}\text{i} \\ -\frac{3}{4} + \frac{2}{7}\text{i} & \frac{1}{4} & -\frac{1}{6}\text{i} & \frac{1}{9} + \frac{1}{7}\text{i} \\ \frac{2}{5} & -\frac{2}{3}\text{i} & \frac{3}{8} + \frac{2}{9}\text{i} & 0 \\ \frac{1}{10} & \frac{5}{8}\text{i} & -\frac{3}{11} + \frac{2}{3}\text{i} & \frac{1}{2} \end{pmatrix}$$

2. Consider the three-qubit state vectors

$$|\psi_1\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0,0,0\rangle_{ABC} + |1,1,1\rangle_{ABC}), \tag{1}$$

$$|\psi_2\rangle_{ABC} = \frac{1}{\sqrt{2}}(|0,0,1\rangle_{ABC} + |0,1,0\rangle_{ABC} + |1,0,0\rangle_{ABC}). \tag{2}$$

Determine the following partial traces.

(a) $\text{Tr}_B\left[|\psi_1\rangle\langle\psi_1|_{ABC}\right]$.

(b) $\text{Tr}_B\left[|\psi_2\rangle\langle\psi_2|_{ABC}\right]$.

(c) $\text{Tr}_A\left[p|\psi_1\rangle\langle\psi_1|_{ABC} + (1-p)|\psi_2\rangle\langle\psi_2|_{ABC}\right]$, where $p \in [0,1]$.

(d) $\text{Tr}_C\left[p|\psi_1\rangle\langle\psi_2|_{ABC} + (1-p)|\psi_2\rangle\langle\psi_1|_{ABC}\right]$, where $p \in [0,1]$.

3. Prove the following identity:

$$\text{Tr}_E[(\mathbb{1}_E \otimes M_S)H_{ES}(\mathbb{1}_E \otimes N_S)] = M_S \,\text{Tr}_E[H_{ES}]N_S, \tag{3}$$

where H_{ES}, M_S, and N_S are arbitrary linear operators.

Part 2 *Measurements.*

1. Consider the following two-qubit density operator ρ_{AB} of Alice and Bob:

$$\rho_{AB} = (1-p)|\Phi^+\rangle\langle\Phi^+|_{AB} + \frac{p}{3}\left(|\Phi^-\rangle\langle\Phi^-|_{AB} + |\Psi^+\rangle\langle\Psi^+|_{AB} + |\Psi^-\rangle\langle\Psi^-|_{AB}\right), \tag{4}$$

where we recall the two-qubit Bell state vectors:

$$|\Phi^\pm\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle \pm |1,1\rangle), \tag{5}$$

$$|\Psi^\pm\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle \pm |1,0\rangle). \tag{6}$$

Suppose that Alice and Bob both measure their qubits with respect to the Pauli-z basis; i.e., the POVM is $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$. What is the probability that they obtain *different* measurement outcomes?

2. Consider the qubit state vectors

$$|\psi_k\rangle = \cos\left(\frac{2\pi k}{5}\right)|0\rangle + \sin\left(\frac{2\pi k}{5}\right)|1\rangle, \quad k \in \{0,1,2,3,4\}. \tag{7}$$

Verify that the set $\left\{\frac{2}{5}|\psi_k\rangle\langle\psi_k|\right\}_{k=0}^{4}$ is a POVM.

3. Consider the three-qubit state vector $|\psi_1\rangle$ in (1), known as the *GHZ state vector*, and define the following set of eight state vectors:

$$|\psi_{z,x_1,x_2}\rangle_{ABC} \coloneqq (Z^z \otimes X^{x_1} \otimes X^{x_2})|\psi_1\rangle_{ABC}, \quad z, x_1, x_2 \in \{0,1\}. \tag{8}$$

Prove that the set $\{|\psi_{z,x_1,x_2}\rangle\}_{z,x_1,x_2\in\{0,1\}}$ is an orthonormal basis for $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. Conclude that the set $\{|\psi_{z,x_1,x_2}\rangle\langle\psi_{z,x_1,x_2}|\}_{z,x_1,x_2\in\{0,1\}}$ is a POVM.

4. Extend the above result to an arbitrary number of qubits. Specifically, define the n-qubit GHZ state vector as follows:

$$|\mathrm{GHZ}_n\rangle \coloneqq \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}), \tag{9}$$

for $n \in \{1, 2, \dots\}$. Then, define

$$|\mathrm{GHZ}_{z,\vec{x}}\rangle \coloneqq (Z^z \otimes X^{\vec{x}})|\mathrm{GHZ}_n\rangle, \tag{10}$$

for $z \in \{0,1\}$ and $\vec{x} \in \{0,1\}^{n-1}$. Prove that the set $\{|\mathrm{GHZ}_{z,\vec{x}}\rangle\}_{z\in\{0,1\},\vec{x}\in\{0,1\}^{n-1}}$ is an orthonormal basis for $(\mathbb{C}^2)^{\otimes n}$, and thereby conclude that the set $\{|\mathrm{GHZ}_{z,\vec{x}}\rangle\langle\mathrm{GHZ}_{z,\vec{x}}|\}_{z\in\{0,1\},\vec{x}\in\{0,1\}^{n-1}}$ is a POVM.

5. (BONUS, 5 points) *Fusing two GHZ state vectors.* Consider a GHZ state of three qubits and a GHZ state of four qubits. We can merge these to create a larger GHZ state of six qubits using the following protocol, which is similar to the teleportation protocol.

   Step 1 Apply the CNOT gate to the third qubit of the first GHZ state (which is the control qubit) and the first qubit of the second GHZ state (which is the target qubit).

   Step 2 Then, measure the target qubit in the $\{|0\rangle, |1\rangle\}$ basis.

   Step 3 If the measurement outcome is "0", do nothing; if the outcome is "1", then apply the Pauli-X gate to all of the remaining qubits of the second GHZ state.

   Prove that, with probability one, this procedure results in the larger GHZ state of six qubits.

Part 3 *Bipartite entanglement.*

1. Let $|\Phi_d\rangle$ be the state vector

$$|\Phi_d\rangle \coloneqq \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k, k\rangle. \tag{11}$$

For every unitary operator U, prove that

$$(U \otimes \overline{U})|\Phi_d\rangle = |\Phi_d\rangle, \tag{12}$$

where $\overline{U}$ denotes the complex conjugate of U.

2. Consider the state vector $|\Psi^-\rangle$ defined in (6). Prove that

$$(U \otimes U)|\Psi^-\rangle\langle\Psi^-|(U \otimes U)^\dagger = |\Psi^-\rangle\langle\Psi^-| \tag{13}$$

for every single-qubit unitary U.

3. By calculating the Schmidt decomposition, and explicitly stating the Schmidt rank, determine whether the following state vectors are entangled.
   (a) $|\psi\rangle_{AB} = \frac{1}{\sqrt{23/2}}\big(|0,0\rangle + \mathrm{i}|0,1\rangle + (\frac{1}{2} - \mathrm{i})|0,2\rangle + (1+\mathrm{i})|1,0\rangle + 2|1,1\rangle + \frac{1}{2}\mathrm{i}|2,0\rangle + (1+1\mathrm{i})|2,2\rangle\big)$.

   (b) $|\psi\rangle_{AB} = \frac{1}{\sqrt{70}}\big(|0,0\rangle + 2|0,1\rangle + 3|0,2\rangle + \mathrm{i}|1,0\rangle + (1+2\mathrm{i})|1,1\rangle + (1+3\mathrm{i})|1,2\rangle + (1+\mathrm{i})|2,0\rangle + (3+2\mathrm{i})|2,1\rangle + (4+3\mathrm{i})|2,2\rangle\big)$.

   (c) $|\psi\rangle_{AB} = \frac{1}{\sqrt{98}}\big(|0,0\rangle + 2|0,1\rangle + 3|0,2\rangle + \mathrm{i}|1,0\rangle + 2\mathrm{i}|1,1\rangle + 3\mathrm{i}|1,2\rangle + (2+\mathrm{i})|2,0\rangle + (4+2\mathrm{i})|2,1\rangle + (6+3\mathrm{i})|2,2\rangle\big)$.

Part 4 *Teleportation.*

1. Finish off the calculation of the teleportation protocol presented in class by determining that every outcome of Alice's measurement occurs with probability $\frac{1}{4}$, and carry out the calculation to determine the conditional state achieved by Bob (before his correction operation) for each of these measurement outcomes.

2. Suppose that instead of a pure state, Alice wishes to teleport a mixed state to Bob. Prove that the teleportation protocol, as presented in class, works even when the state to be teleported is a mixed state.