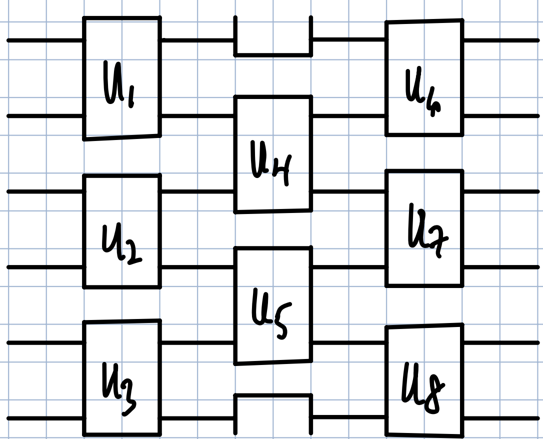
① Random Quantum Circuits

- Random unitaries are those drawn from the Haar measure (think: "uniform distribution" over unitaries)

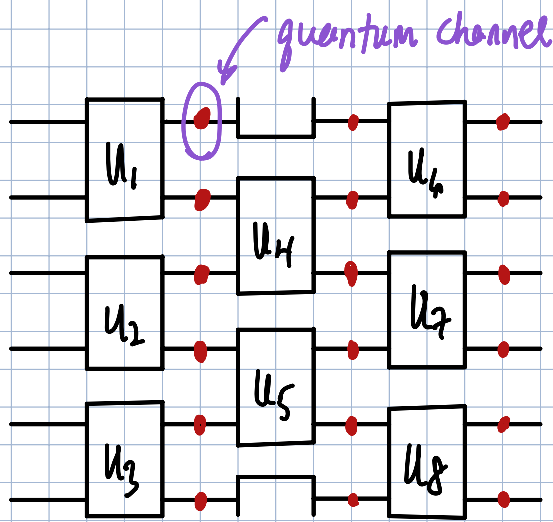
- Used in physics to model chaotic dynamics / "scrambling".
- Used in CS context to assess quantum advantage, in cryptography (as a source of randomness)

- How "complex" are random n-qubit unitaries?
  ↳ How can we generate a Haar random unitary from more elementary gates? (e.g., Clifford gates?).
  ↳ or a k-design unitary.

- Look at random quantum circuits



(Noiseless)



(Noisy)

⊛ Pick every two-qubit gate at random from the Haar distribution. (or some other ensemble).

Near-linear constructions of exact unitary 2-designs

arXiv:1501.04592

Richard Cleve*† Debbie Leung‡† Li Liu* Chunhao Wang*

We give three constructions of *exact* unitary 2-designs on n qubits that have the following quantum gate costs (number of one- and two-qubit gates):

- $O(n \log n \log\log n)$ gates (all Clifford gates) for infinitely many n, assuming the extended Riemann Hypothesis is true.
- $O(n \log n \log\log n)$ gates (including non-Clifford gates) for all n, unconditionally.
- $O(n \log^2 n \log\log n)$ gates (all Clifford gates) for all n, unconditionally.

The circuits for the first two constructions can be organized so as to perform their computation in $O(\log n)$ depth; the third in $O(\log^2 n)$-depth (using the fact that efficient multiplication/convolution algorithms require only $O(\log n)$-depth [33]). These results are near optimal – in Appendix E, we show that for any unitary 2-design (exact or approximate under Definition 2 or 3), a high probability set of the unitaries have size $\Omega(n)$ and depth $\Omega(\log n)$.

⊛ Recall: a unitary 2-design is any set $\{U_j\}_{j=1}^{N}$ such that

$$\mathbb{E}_{\underset{U\sim\text{Haar}}{U}}\left[U^{\otimes 2} M U^{\dagger\otimes 2}\right] = \frac{1}{N}\sum_{j=1}^{N} U_j^{\otimes 2} M U_j^{\dagger\otimes 2}$$

Random unitaries in extremely low depth

arXiv:2407.07754

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⊛ Recall: a unitary k-design is any set $\{U_j\}_{j=1}^{N}$ such that

$$\mathbb{E}_U\left[U^{\otimes k} M U^{\dagger\otimes k}\right] = \frac{1}{N}\sum_{j=1}^{N} U_j^{\otimes k} M U_j^{\dagger\otimes k}$$

Corollary 1 (Low-depth random unitary designs). *Random quantum circuits over n qubits can form ε-approximate unitary k-designs in circuit depth*

- $d = \mathcal{O}(\log(n/\varepsilon) \cdot k \operatorname{poly}\log(k))$, *for 1D circuits without ancilla qubits,*
- $d = \mathcal{O}(\log\log(n/\varepsilon))$, *for all-to-all circuits with $\mathcal{O}(n\log(n/\varepsilon))$ ancilla qubits and $k \leq 3$.*

APPROXIMATE UNITARY k-DESIGNS FROM SHALLOW, LOW-COMMUNICATION CIRCUITS

arXiv:2407.07876

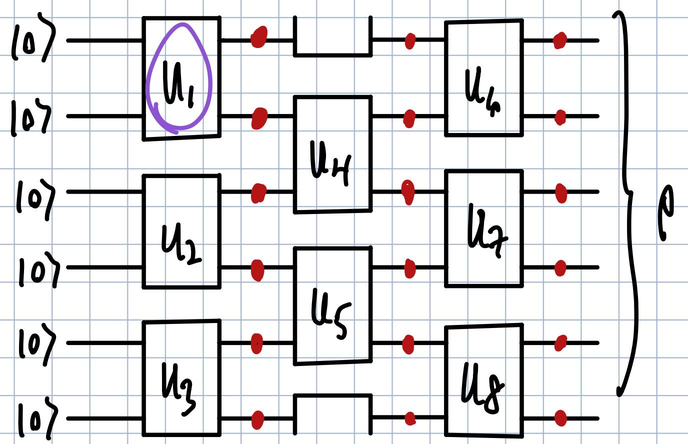
NICHOLAS LARACUENTE AND FELIX LEDITZKY

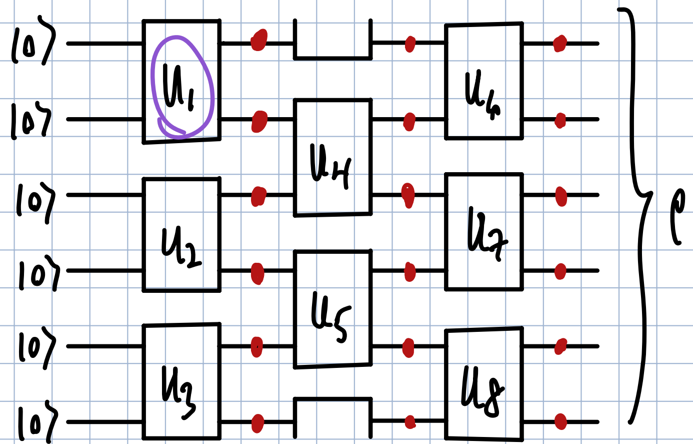
We give a construction of an ϵ-approximate relative k-design depth

$$O\left(k(k\log k + \log \overset{n}{m} + \log(1/\epsilon))\operatorname{polylog}(k)\right) . \qquad (1.3)$$

- Consider the noisy case (noiseless is a special case).
We focus on quantities that require access to only two copies of the circuit — e.g., purity.

↳ Recall: $\mathrm{Tr}[\rho^2] = \mathrm{Tr}[\underset{\text{swap operator.}}{F}(\rho\otimes\rho)]$.





- Each U_i is chosen independently.
- We have two copies of each unitary.
- We can take averages independently over every two-qubit pair.

(Recall: If X and Y are indep. RVs, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.)

- We have to calculate the expectation values $\boxed{\mathbb{E}_{U_i}\left[(U_i \otimes U_i)\,\rho\,(U_i \otimes U_i)^\dagger\right]}$ over every qubit pair

⊛ B/c we need only two copies, the circuit can consist of randomly-chosen Clifford gates! (B/c the Clifford gates form a unitary 2-design.)

⊛ How does purity decay as a function of depth?

↳ This can also be used to address entanglement growth in the circuit.

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Quantum Entanglement Growth under Random Unitary Dynamics

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PHYSICAL REVIEW B **99**, 174205 (2019)

Editors' Suggestion

Emergent statistical mechanics of entanglement in random unitary circuits

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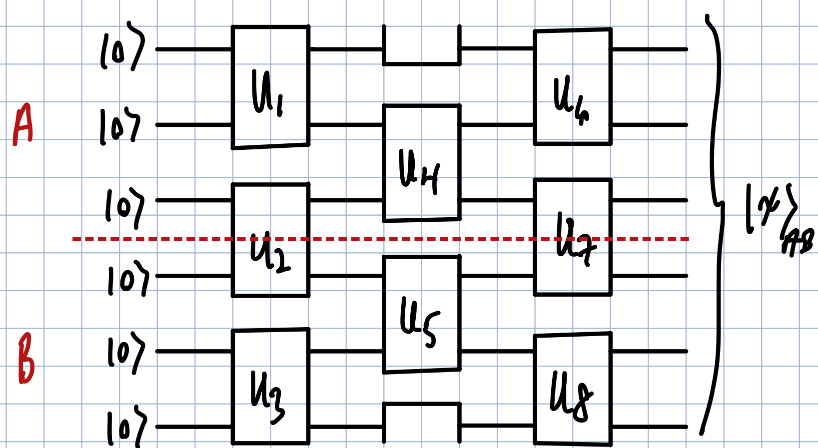


⊛ We can quantify the entanglement using <u>entanglement entropy</u>:

$$E(|\psi\rangle) = S(\mathrm{Tr}_B[|\psi\rangle\langle\psi|_{AB}])$$

$$S(\rho) = -\mathrm{Tr}[\rho \log \rho]. \quad \text{(entropy)}$$

⊛ $S(\rho) \geq \underbrace{S_2(\rho) = -\log \mathrm{Tr}[\rho^2]}_{\text{Renyi-2 entropy}}$ purity!

(aka "collision entropy")

$$\Rightarrow \underline{\underline{E(|\psi\rangle_{AB}) \geq -\log \mathrm{Tr}[\rho_A^2]}}$$

$$|\psi\rangle_{AB} = \underset{\text{unitary for the full circuit}}{U}|0\rangle^{\otimes n} \rightarrow \rho_A = \mathrm{Tr}_B[U|0\rangle\langle 0|^{\otimes n} U^\dagger]$$

$$\mathrm{Tr}[\rho_A^2] = \mathrm{Tr}[F(\rho_A \otimes \rho_A)] = \mathrm{Tr}[(F_{A_1A_2} \otimes \mathbb{1}_{B_1B_2})(|\psi\rangle\langle\psi|_{A_1B_1} \otimes |\psi\rangle\langle\psi|_{A_2B_2})]$$

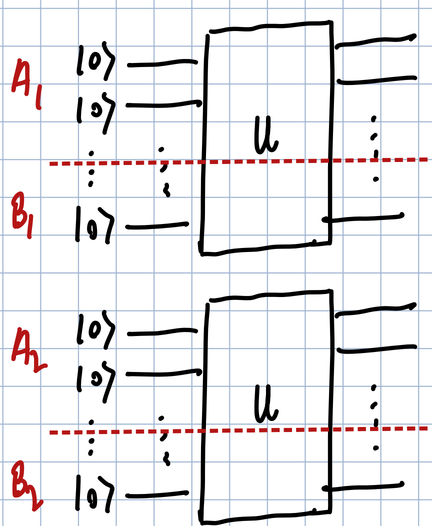
$$= \mathrm{Tr}[(F_{A_1A_2} \otimes \mathbb{1}_{B_1B_2})((U\otimes U)|0,0\rangle\langle 0,0|^{\otimes n}(U\otimes U)^\dagger)]$$

$$\Rightarrow \mathbb{E}_U[\mathrm{Tr}[\rho_A^2]] = \mathrm{Tr}[(F_{A_1A_2} \otimes \mathbb{1}_{B_1B_2})\, \underline{\underline{\mathbb{E}_U[(U\otimes U)|0,0\rangle\langle 0,0|^{\otimes n}(U\otimes U)^\dagger]}}] \quad \alpha\mathbb{1} + \beta F$$

$$\boxed{\Rightarrow \mathbb{E}_U[E(|\psi\rangle_{AB})] \geq -\log_2 \mathbb{E}_U[\mathrm{Tr}[\rho_A^2]]}$$ → Lower bound on entanglement entropy of a random pure state.

② <u>Warm-Up: Global Haar-Random Evolution</u>

<u>(a)</u>



$d = 2^n$, $d_A = 2^k$, $d_B = 2^{n-k}$

- Look at noiseless case first.
- Consider two copies and take the average.

$$\mathbb{E}_U[(U\otimes U)(\rho\otimes\rho)(U\otimes U)^\dagger]$$

$\rho\otimes\rho$: initial state

- From last time, we know that

$$\mathbb{E}_U[(U\otimes U)(\rho\otimes\rho)(U\otimes U)^\dagger] = \alpha\mathbb{1}\otimes\mathbb{1} + \beta F$$

$$\alpha = \frac{1}{d^2-1}\left(\mathrm{Tr}[\underbrace{M}_{\rho\otimes\rho}] - \frac{1}{d}\mathrm{Tr}[F\underbrace{M}_{\rho\otimes\rho}]\right) = \frac{1}{d^2-1}\left(\underbrace{\mathrm{Tr}[\rho\otimes\rho]}_{=\mathrm{Tr}[\rho]^2 = 1} - \frac{1}{d}\underbrace{\mathrm{Tr}[F(\rho\otimes\rho)]}_{=\mathrm{Tr}[\rho^2]}\right) = \frac{1}{d^2-1}\left(1 - \frac{1}{d}\mathrm{Tr}[\rho^2]\right)$$

$$= \frac{1}{d^2-1}\left(1-\frac{1}{d}\right) \quad \text{(If initial state is pure).}$$

$$= \frac{1}{d(d+1)}$$

$$\beta = \frac{1}{d^2-1}\left(\mathrm{Tr}[F\underbrace{M}_{\rho\otimes\rho}] - \frac{1}{d}\mathrm{Tr}[\underbrace{M}_{\rho\otimes\rho}]\right) = \frac{1}{d^2-1}\left(\mathrm{Tr}[\rho^2] - \frac{1}{d}\right) = \frac{1}{\underbrace{d^2-1}_{(d-1)(d+1)}}\underbrace{\left(1-\frac{1}{d}\right)}_{\frac{d-1}{d}} = \frac{1}{d(d+1)}$$

- Now, from above, $\mathbb{E}_U[\mathrm{Tr}[\rho_A^2]] = \mathrm{Tr}\left[(F_{A_1A_2}\otimes \mathbb{1}_{B_1B_2})\, \underbrace{\mathbb{E}_U[(U\otimes U)|0,0\rangle\langle 0,0|^{\otimes n}(U\otimes U)^\dagger]}_{\frac{1}{d(d+1)}(\mathbb{1}_{A_1A_2}\otimes\mathbb{1}_{B_1B_2} + F_{A_1A_2}\otimes F_{B_1B_2})}\right]$

$$F_{A_1A_2} = \sum_{i,j=0}^{d_A-1} |i,j\rangle\langle j,i| \to \mathrm{Tr}[F] = \sum_{i,j=0}^{d-1}\delta_{i,j} = d_A$$

$$= \frac{1}{d(d+1)}\mathrm{Tr}[F_{A_1A_2}\otimes\mathbb{1}_{B_1B_2} + \underbrace{F^2_{A_1A_2}}_{=\mathbb{1}_{A_1A_2}}\otimes F_{B_1B_2}]$$

$$= \frac{1}{d(d+1)}\left(\underbrace{\mathrm{Tr}[F_{A_1A_2}]}_{=d_A}\underbrace{\mathrm{Tr}[\mathbb{1}_{B_1B_2}]}_{d_B^2} + \underbrace{\mathrm{Tr}[\mathbb{1}_{A_1A_2}]}_{d_A^2}\underbrace{\mathrm{Tr}[F_{B_1B_2}]}_{d_B}\right) \qquad (d \equiv d_A d_B)$$

$$= \frac{1}{d_Ad_B(d_Ad_B+1)}\left(d_Ad_B^2 + d_A^2 d_B\right) = \frac{d_Ad_B}{d_Ad_B(d_Ad_B+1)}\left(d_B+d_A\right)$$

$$= \frac{d_A+d_B}{d_Ad_B+1} \qquad (\text{e.g., } d_A = 2^k,\ d_B = 2^{n-k})$$

$$\Rightarrow \mathbb{E}_U[E(|\psi\rangle_{AB})] \geq -\log_2\left[\frac{d_A+d_B}{d_Ad_B+1}\right]$$

⊛ Note: Maximum value of $E(|\psi\rangle_{AB})$ is $\log_2 d_A$ $\left(S(\rho_A) \leq \log_2 d_A \ \forall \rho_A\right)$.

e.g., for $|\psi\rangle_{AB} = |\Phi^+\rangle_{AB}$ (n-qubit maximally-entangled state).

$$\rho_A = \frac{\mathbb{1}}{2^n} \Rightarrow E(|\Phi^+\rangle_{AB}) = \log_2 2^n = n.$$

⊛ Note: Exact value of $\mathbb{E}_U[E(|\psi\rangle_{AB})]$ is known! Called the **Page Entropy**.

For $d_A \le d_B$, $$\boxed{\mathbb{E}_U[E(|\psi\rangle_{AB})] = \frac{1}{\ln(2)}\left(\sum_{k=d_B+1}^{d_A d_B}\frac{1}{k} - \frac{d_A-1}{2d_B}\right)}$$

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Average Entropy of a Subsystem

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(Received 7 May 1993)

If a quantum system of Hilbert space dimension mn is in a random pure state, the average entropy of a subsystem of dimension $m \le n$ is conjectured to be $S_{m,n} = \sum_{k=n+1}^{mn} \frac{1}{k} - \frac{m-1}{2n}$ and is shown to be $\simeq \ln m - \frac{m}{2n}$ for $1 \ll m \le n$. Thus there is less than one-half unit of information, on average, in the smaller subsystem of a total system in a random pure state.

PHYSICAL REVIEW LETTERS

VOLUME 77 | 1 JULY 1996 | NUMBER 1

Average Entropy of a Quantum Subsystem

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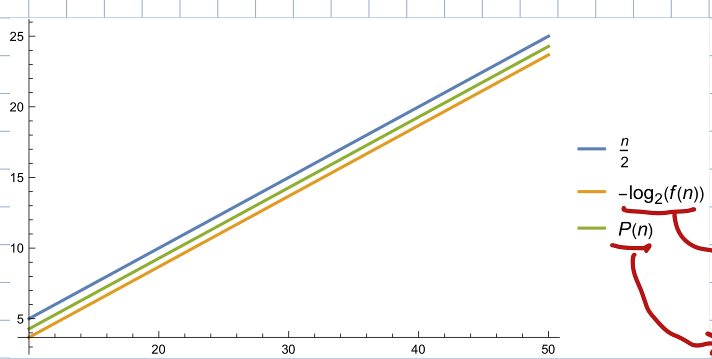
(Received 24 January 1996)

It was recently conjectured by D. Page that if a quantum system of Hilbert space dimension nm is in a random pure state then the average entropy of a subsystem of dimension m where $m \le n$ is $S_{m,n} = (\sum_{k=n+1}^{mn} 1/k) - (m-1)/2n$. In this Letter a simple proof of this conjecture is given. [S0031-9007(96)00569-8]



$d_A = d_B = 2^{n/2}$ ($n \equiv$ total # of qubits)

Renyi-2 lower bound

Page entropy.

Commun. Math. Phys. 265, 95–117 (2006)
Digital Object Identifier (DOI) 10.1007/s00220-006-1535-6

Communications in **Mathematical Physics**

Aspects of Generic Entanglement

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Theorem III.3 (Concentration of entropy). *Let $\varphi \in_R \mathcal{P}(A \otimes B)$ be a random state on $A \otimes B$, with $d_B \geq d_A \geq 3$. Then*

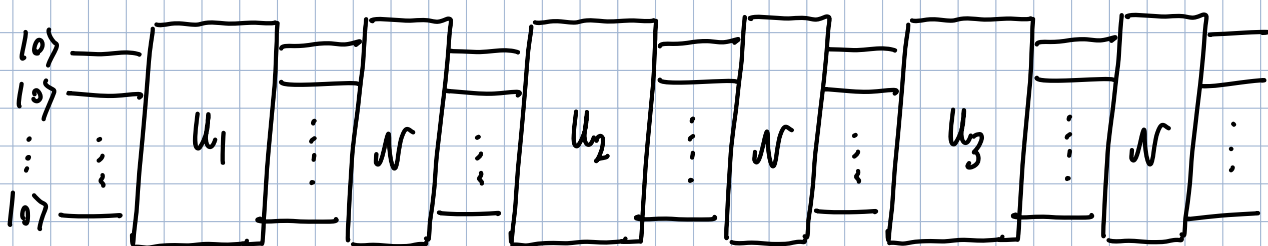
$$\Pr\{\underbrace{S(\varphi_A)}_{= E(|\psi\rangle_{AB})} < \log d_A - \alpha - \beta\} \leq \exp\left(-\frac{(d_A d_B - 1)C_3\alpha^2}{(\log d_A)^2}\right),$$

where $\beta = \frac{1}{\ln 2}\frac{d_A}{d_B}$ is as in Lemma II.4 and $C_3 = (8\pi^2 \ln 2)^{-1}$.

$$d_A = d_B = 2^{n/2} \Rightarrow \Pr\left[E(|\psi\rangle_{AB}) < \frac{n}{2} - \alpha - \frac{1}{\ln(2)}\right] \leq \exp\left(-\frac{(2^n-1)C_3\alpha^2}{n^2/4}\right) \to 0 \text{ as } n \to \infty$$

$\Rightarrow$ With high prob. (as $n \to \infty$), a random pure state is maximally entangled!

(b) Now add noise!



– Let ρ be an initial state. The state after L layers is

$$\rho(L) = (\underbrace{\mathcal{N} \circ \mathcal{U}_L}_{\mathcal{M}_L} \circ \underbrace{\mathcal{N} \circ \mathcal{U}_{L-1}}_{\mathcal{M}_{L-1}} \circ \cdots \circ \underbrace{\mathcal{N} \circ \mathcal{U}_2}_{\mathcal{M}_2} \circ \underbrace{\mathcal{N} \circ \mathcal{U}_1}_{\mathcal{M}_1})(\rho) \qquad \mathcal{U}_i(\cdot) = U_i \rho U_i^\dagger$$

We consider the purity of $\rho(L)$: $\mathrm{Tr}[F(\rho(L) \otimes \rho(L))] = \mathrm{Tr}[(\rho(L))^2]$.

$$\mathbb{E}_{U_1,U_2,\dots,U_L}\left[\mathrm{Tr}\left[(\rho(L))^2\right]\right] = \mathbb{E}_{U_1,U_2,\dots,U_L}\left[\mathrm{Tr}\left[F(\rho(L)\otimes\rho(L))\right]\right]$$

$$\rho(L) = \underbrace{(\mathcal{M}_L\circ\mathcal{M}_{L-1}\circ\cdots\circ\mathcal{M}_2\circ\mathcal{N})}_{\mathcal{M}_{2:L}}(\mathcal{U}_1(\rho))$$

$$= \mathbb{E}_{U_1,\dots,U_L}\left[\mathrm{Tr}\left[F\left(\mathcal{M}_{2:L}(\mathcal{U}_1(\rho))\otimes\mathcal{M}_{2:L}(\mathcal{U}_1(\rho))\right)\right]\right]$$

$$= \mathbb{E}_{U_2,\dots,U_L}\left[\mathrm{Tr}\left[F\left((\mathcal{M}_{2:L}\otimes\mathcal{M}_{2:L})\,\underline{\mathbb{E}_{U_1}\left[(\mathcal{U}_1\otimes\mathcal{U}_1)(\rho\otimes\rho)\right]}\right)\right]\right]$$

$$= \mathbb{E}_{U_1}\left[(U_1\otimes U_1)(\rho\otimes\rho)(U_1\otimes U_1)^\dagger\right]$$

(⊛ Each U_i chosen <u>independently</u> at random!)

$$-\ \mathcal{T}(\rho\otimes\rho) = \mathbb{E}_U\left[(U\otimes U)(\rho\otimes\rho)(U\otimes U)^\dagger\right] = \alpha\frac{\mathbb{1}\otimes\mathbb{1}}{d^2} + \beta\frac{F}{d}$$

$$\alpha = \frac{1}{d^2-1}\left(1-\frac{1}{d}\mathrm{Tr}[\rho^2]\right)$$

$$\beta = \frac{1}{d^2-1}\left(\mathrm{Tr}[\rho^2]-\frac{1}{d}\right)$$

$$= p_0\underbrace{\frac{\mathbb{1}\otimes\mathbb{1}}{d^2}}_{M_0} + p_1\underbrace{\frac{F}{d}}_{M_1}$$

$$p_0 = d^2\alpha = \frac{d^2}{d^2-1}\left(1-\frac{1}{d}\mathrm{Tr}[\rho^2]\right),\quad p_1 = d\beta = \frac{d}{d^2-1}\left(\mathrm{Tr}[\rho^2]-\frac{1}{d}\right)$$

- B/c $\frac{1}{d}\leq\mathrm{Tr}[\rho^2]\leq 1$, $p_0\geq 0$ and $p_1\geq 0$.

- Also: $p_0+p_1 = \frac{d}{d^2-1}\left(d-\cancel{\mathrm{Tr}[\rho^2]}+\cancel{\mathrm{Tr}[\rho^2]}-\frac{1}{d}\right) = \frac{d}{d^2-1}\left(d-\frac{1}{d}\right) = \frac{d}{d^2-1}\left(\frac{d^2-1}{d}\right) = 1$

$\Rightarrow \{p_0, p_1\}$ is a <u>probability distribution</u>!

• Let $\vec{p}_0 = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$, $\mathrm{Tr}[F\,\mathcal{J}(\rho\otimes\rho)] = \frac{1}{d} p_0 + d\, p_1 = \mathrm{Tr}[\rho^2]$.

– Back to the purity calculation:

$$\mathbb{E}_{U_1,U_2,\dots,U_L}\left[\mathrm{Tr}\left[(\rho(L))^2\right]\right] = \mathbb{E}_{U_2,\dots,U_L}\left[\mathrm{Tr}\left[F\left((\mathcal{M}_{2:L}\otimes\mathcal{M}_{2:L})\,\underline{\mathbb{E}_{U_1}\left[(\mathcal{U}_1\otimes\mathcal{U}_1)(\rho\otimes\rho)\right]}\right)\right]\right]$$

$$= \mathbb{E}_{U_1}\left[(U_1\otimes U_1)(\rho\otimes\rho)(U_1\otimes U_1)^\dagger\right]$$

$\frac{1}{d} p_0 + d\, p_1$

$$= \mathbb{E}_{U_2,\dots,U_L}\left[\mathrm{Tr}\left[F\left((\underbrace{\mathcal{M}_{2:L}}\otimes\underbrace{\mathcal{M}_{2:L}}_{=\mathcal{M}_{3:L}\circ\,\mathcal{U}_2\circ\mathcal{N}})(p_0 M_0 + p_1 M_1)\right)\right]\right]$$

$$= \mathbb{E}_{U_2,\dots,U_L}\left[\mathrm{Tr}\left[F\left((\mathcal{M}_{3:L}\otimes\mathcal{M}_{3:L})(\mathcal{U}_2\otimes\mathcal{U}_2)\left(p_0(\mathcal{N}\otimes\mathcal{N})(M_0) + p_1(\mathcal{N}\otimes\mathcal{N})(M_1)\right)\right)\right]\right]$$

$$= \mathbb{E}_{U_3,\dots,U_L}\left[\mathrm{Tr}\left[F\left((\mathcal{M}_{3:L}\otimes\mathcal{M}_{3:L})\,\underline{\underline{\mathbb{E}_{U_2}\left[(\mathcal{U}_2\otimes\mathcal{U}_2)\left(p_0(\mathcal{N}\otimes\mathcal{N})(M_0) + p_1(\mathcal{N}\otimes\mathcal{N})(M_1)\right)\right]}}\right)\right]\right]$$

$$\alpha\, \mathbb{1}\otimes\mathbb{1} + \beta F = d^2\alpha M_0 + d\beta M_1$$

Do it term by term:

• $\mathbb{E}_{U_2}\left[(U_2\otimes U_2)\,\underbrace{(\mathcal{N}\otimes\mathcal{N})(M_0)}\,(U_2\otimes U_2)^\dagger\right] = \overbrace{d^2\alpha_{00}}^{t_{00}} M_0 + \overbrace{d\beta_{10}}^{t_{10}} M_1$

$$\alpha_{00} = \frac{1}{d^2-1}\left(\underbrace{\mathrm{Tr}[(\cdot)]}_{①} - \frac{1}{d}\underbrace{\mathrm{Tr}[F(\cdot)]}_{②}\right) \qquad \beta_{10} = \frac{1}{d^2-1}\left(\underbrace{\mathrm{Tr}[F(\cdot)]}_{②} - \frac{1}{d}\underbrace{\mathrm{Tr}[(\cdot)]}_{①}\right)$$

$$①: \mathrm{Tr}[(\mathcal{N}\otimes\mathcal{N})(M_0)] = \mathrm{Tr}[M_0] \quad (\text{If } \mathcal{N} \text{ is trace preserving!})$$
$$= 1 \qquad \left(M_0 = \frac{\mathbb{1}\otimes\mathbb{1}}{d^2}\right)$$

$$②: \mathrm{Tr}[F(\mathcal{N}\otimes\mathcal{N})(M_0)] = \frac{1}{d^2}\mathrm{Tr}[F(\mathcal{N}\otimes\mathcal{N})(\mathbb{1}\otimes\mathbb{1})]$$
$$= \frac{1}{d^2}\mathrm{Tr}[F(\mathcal{N}(\mathbb{1})\otimes\mathcal{N}(\mathbb{1}))]$$
$$= \frac{1}{d^2}\mathrm{Tr}[(\mathcal{N}(\mathbb{1}))^2]$$
$$= \mathrm{Tr}\left[\left(\mathcal{N}\left(\frac{\mathbb{1}}{d}\right)\right)^2\right]$$
$$\equiv \mu(\mathcal{N}). \rightarrow \text{"unitality" of } \mathcal{N}.$$

$\mathcal{N}$ unital $\rightarrow \mathcal{N}(\mathbb{1}) = \mathbb{1}$

$\mu(\mathcal{N}) = \mathrm{Tr}\left[\left(\frac{\mathbb{1}}{d}\right)^2\right] = \frac{1}{d}$

$\alpha\,\mathbb{1}\otimes\mathbb{1} + \beta F$

$$\bullet\ \mathbb{E}_{U_2}\left[(U_2\otimes U_2)(\mathcal{N}\otimes\mathcal{N})(M_1)(U_2\otimes U_2)^\dagger\right] = d^2\overbrace{\alpha_{01}}^{t_{01}} M_0 + d\overbrace{\beta_{11}}^{t_{11}} M_1$$

$M_1 = \frac{F}{d}$

$$\alpha_{01} = \frac{1}{d^2-1}\left(\underbrace{\mathrm{Tr}[(\cdot)]}_{③} - \frac{1}{d}\underbrace{\mathrm{Tr}[F(\cdot)]}_{④}\right) \qquad \beta_{11} = \frac{1}{d^2-1}\left(\underbrace{\mathrm{Tr}[F(\cdot)]}_{④} - \frac{1}{d}\underbrace{\mathrm{Tr}[(\cdot)]}_{③}\right)$$

$$③: \mathrm{Tr}[(\mathcal{N}\otimes\mathcal{N})(M_1)] = \mathrm{Tr}[M_1] = 1 \quad (\text{b/c } \mathcal{N} \text{ TP!}) \quad \left(M_1 = \frac{1}{d}F\right)$$

$\mathrm{Tr}[Y\underline{\mathcal{N}}(X)] = \mathrm{Tr}[\mathcal{N}^\dagger(Y)X]$

$$④: \mathrm{Tr}[F(\mathcal{N}\otimes\mathcal{N})(M_1)] = \frac{1}{d}\mathrm{Tr}[F(\mathcal{N}\otimes\mathcal{N})(F)] \equiv d\,\eta(\mathcal{N}).$$
$$= \frac{1}{d}\mathrm{Tr}[\underbrace{(\mathcal{N}^\dagger\otimes \mathrm{id})(F)}(\mathrm{id}\otimes\mathcal{N})(F)] = \frac{1}{d}\mathrm{Tr}[((\mathrm{id}\otimes\mathcal{N})(F))^2]$$
$$\hookrightarrow = (\mathrm{id}\otimes\mathcal{N})(F)$$

$$\left((\mathbb{1}\otimes M)|\Phi^+\rangle = (M^T\otimes\mathbb{1})|\Phi^+\rangle\right)$$

Choi rep.

$$\mathcal{N}_{A\to B} \rightarrow C^{\mathcal{N}}_{AB} = \sum_{i,j=0}^{d-1} |i\rangle\langle j| \otimes \mathcal{N}(|j\rangle\langle i|)$$
$$= d\,\underbrace{(\mathrm{id}\otimes\mathcal{N})(\Phi^+)}_{\text{Choi state}}$$

$$\mathrm{Tr}[(\mathrm{id}_A\otimes\mathcal{N}_B)(\Phi^+)]$$
$$= \mathrm{Tr}_A\mathrm{Tr}_B[(\mathrm{id}_A\otimes\mathcal{N}_B)(\Phi^+)]$$

$\mathrm{Tr}_B[\Phi^+] = \frac{\mathbb{1}_A}{d_A}$

Proof: $(\mathcal{N}^\dagger\otimes \mathrm{id})(F) = \sum_{i,j=0}^{d-1} \mathcal{N}^\dagger(|j\rangle\langle i|)\otimes|i\rangle\langle j|$

$$\Rightarrow (\langle k|\otimes\mathbb{1})(\mathcal{N}^\dagger\otimes \mathrm{id})(F)(|l\rangle\otimes\mathbb{1}) = \sum_{i,j=0}^{d-1}\langle k|\mathcal{N}^\dagger(|j\rangle\langle i|)|l\rangle\,|i\rangle\langle j|$$
$$= \sum_{i,j=0}^{d-1}\mathrm{Tr}[|l\rangle\langle k|\mathcal{N}^\dagger(|j\rangle\langle i|)]\,|i\rangle\langle j|$$

$$= \mathrm{Tr}_A\left[\frac{\mathbb{1}}{d_A}\right] = 1 \checkmark$$

$$\eta(\mathcal{N}) = \frac{1}{d^2}\mathrm{Tr}\left[\left((id\otimes\mathcal{N})(d\Phi^+)\right)^2\right]$$
$$= \mathrm{Tr}\Big[\big(\underbrace{(id\otimes\mathcal{N})(\Phi^+)}_{\text{Choi state!}}\big)^2\Big]$$

$$F^{T_A} = \sum_{i,j=0}^{d-1} (|i\rangle\langle j|)^T \otimes |j\rangle\langle i|$$
$$= \sum_{i,j=0}^{d-1} |j\rangle\langle i| \otimes |j\rangle\langle i|$$
$$= \sum_{i,j=0}^{d-1} |j,j\rangle\langle i,i| = d\,\Phi^+$$

$$= \sum_{i,j=0}^{d-1} \mathrm{Tr}\left[\mathcal{N}(|l\rangle\langle k|)\,|j\rangle\langle i|\right] |i\rangle\langle j|$$
$$= \sum_{i,j=0}^{d-1} |i\rangle\langle i|\,\mathcal{N}(|l\rangle\langle k|)\,|j\rangle\langle j|$$
$$= \mathcal{N}(|l\rangle\langle k|)$$

$$\Rightarrow (\mathcal{N}^\dagger\otimes id)(F) = \sum_{k,l=0}^{d-1} |k\rangle\langle l| \otimes \left(\langle k|\otimes\mathbb{1}\right)(\mathcal{N}^\dagger\otimes id)(F)\left(|l\rangle\otimes\mathbb{1}\right)$$
$$= \sum_{k,l=0}^{d-1} |k\rangle\langle l| \otimes \mathcal{N}(|l\rangle\langle k|) = (id\otimes\mathcal{N})(F). \quad \blacksquare$$

- Put it all together:

$$\mathbb{E}_{\mathcal{U}_2}\left[(\mathcal{U}_2\otimes\mathcal{U}_2)\left(p_0(\mathcal{N}\otimes\mathcal{N})(M_0) + p_1(\mathcal{N}\otimes\mathcal{N})(M_1)\right)\right] = p_0\left(t_{00}M_0 + t_{10}M_1\right) + p_1\left(t_{01}M_0 + t_{11}M_1\right)$$
$$= \left(p_0t_{00} + p_1t_{01}\right)M_0 + \left(p_0t_{10} + p_1t_{11}\right)M_1$$

- Let
$$\vec{p}_1 = \begin{pmatrix} p_0t_{00} + p_1t_{01} \\ p_0t_{01} + p_1t_{11} \end{pmatrix} = \underbrace{\begin{pmatrix} t_{00} & t_{01} \\ t_{10} & t_{11} \end{pmatrix}}_{\equiv T} \underbrace{\begin{pmatrix} p_0 \\ p_1 \end{pmatrix}}_{\vec{p}_0}$$

$$t_{00} = \frac{d^2}{d^2-1}\left(1 - \frac{\mu(\mathcal{N})}{d}\right), \quad t_{10} = \frac{d}{d^2-1}\left(\mu(\mathcal{N}) - \frac{1}{d}\right), \quad \mu(\mathcal{N}) = \mathrm{Tr}\left[\left(\mathcal{N}\left(\frac{\mathbb{1}}{d}\right)\right)^2\right]$$

$$t_{01} = \frac{d^2}{d^2-1}\left(1 - \eta(\mathcal{N})\right), \quad t_{11} = \frac{d}{d^2-1}\left(d\,\eta(\mathcal{N}) - \frac{1}{d}\right), \quad \eta(\mathcal{N}) = \frac{1}{d^2}\mathrm{Tr}\left[\left((id\otimes\mathcal{N})(F)\right)^2\right]$$
$$= \mathrm{Tr}\left[\left((id\otimes\mathcal{N})(\Phi^+)\right)^2\right].$$

- Both $\mu(\mathcal{N})$ and $\eta(\mathcal{N})$ are purities $\Rightarrow \frac{1}{d} \le \mu(\mathcal{N}) \le 1, \quad \frac{1}{d^2} \le \eta(\mathcal{N}) \le 1$

$\Rightarrow t_{00}, t_{01}, t_{10}, t_{11} \geq 0.$

- Also, $t_{00} + t_{10} = \frac{d}{d^2-1}\left(d - \cancel{\mu(\mathcal{N})} + \cancel{\mu(\mathcal{N})} - \frac{1}{d}\right) = \frac{d}{d^2-1}\,\frac{d^2-1}{d} = 1$

$$t_{01} + t_{11} = \frac{d}{d^2-1}\left(d - d\cancel{\eta(\mathcal{N})} + d\cancel{\eta(\mathcal{N})} - \frac{1}{d}\right) = \frac{d}{d^2-1}\,\frac{d^2-1}{d} = 1$$

$\Rightarrow T = \begin{pmatrix} t_{00} & t_{01} \\ t_{10} & t_{11} \end{pmatrix}$ is a stochastic matrix (columns form probability distributions)

- Conclude purity calculation:

$$\mathbb{E}_{U_1,U_2,\dots,U_L}\left[\mathrm{Tr}\left[(\rho(L))^2\right]\right] = \mathbb{E}_{U_1,U_2,\dots,U_L}\left[\mathrm{Tr}\left[F(\rho(L)\otimes\rho(L))\right]\right]$$

$$= \mathbb{E}_{U_3,\dots,U_L}\left[\mathrm{Tr}\left[F\left((M_{3:L}\otimes M_{3:L})\underbrace{\mathbb{E}_{U_2}\left[(U_2\otimes U_2)\left(p_0(\mathcal{N}\otimes\mathcal{N})(M_0) + p_1(\mathcal{N}\otimes\mathcal{N})(M_1)\right)\right]}_{(p_0t_{00}+p_1t_{01})M_0 + (p_0t_{10}+p_1t_{11})M_1}\right)\right]\right]$$

$\vec{p}_0 = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$

$\vec{p}_1 = T\vec{p}_0 \rightarrow \vec{p}_2 = T^2\vec{p}_0$

⊛ We will always have a linear combination of M_0 and M_1!

⊛ Probability vector for the final state $\rho(L)$ will be $\underline{\underline{\vec{p}_L = T^L\vec{p}_0}} \equiv \begin{pmatrix} p_{L,0} \\ p_{L,1} \end{pmatrix}$

$\Rightarrow$ Markov chain structure! The Markov chain has only <u>two</u> states, and the state transitions are governed by the matrix T.

⊛ Recover purity from $\vec{p}_L$: $\mathrm{Tr}\left[(\rho(L))^2\right] = \frac{1}{d}p_{L,0} + d\,p_{L,1}$

• Now $T = \begin{pmatrix} a & 1-b \\ 1-a & b \end{pmatrix} \Rightarrow T^L = \begin{pmatrix} a_L & 1-b_L \\ 1-a_L & b_L \end{pmatrix}$

$$\begin{pmatrix} a = \frac{d^2}{d^2-1}\left(1 - \frac{\mu(W)}{d}\right) \\ b = \frac{d}{d^2-1}\left(d\eta(W) - \frac{1}{d}\right) \end{pmatrix} \qquad \begin{aligned} a_L &= 1 - \left(\frac{1-c^L}{1-c}\right)(1-a) \\ b_L &= 1 - \left(\frac{1-c^L}{1-c}\right)(1-b), \quad c = -1+a+b. \end{aligned}$$

$$\Rightarrow \vec{p}_L = \begin{pmatrix} p_{0,L} \\ p_{1,L} \end{pmatrix} = \begin{pmatrix} a_L & 1-b_L \\ 1-a_L & b_L \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$$

$$\left.\begin{aligned} p_0 &= \frac{d^2}{d^2-1}\left(1 - \frac{1}{d}\mathrm{Tr}[\rho^2]\right) = \frac{d}{d+1} \\ p_1 &= \frac{d}{d^2-1}\left(\underbrace{\mathrm{Tr}[\rho^2]}_{\text{initial state}} - \frac{1}{d}\right) = \frac{1}{d+1} \end{aligned}\right\} \text{If initial state pure.}$$

$$= \begin{pmatrix} a_L p_0 + (1-b_L)p_1 \\ (1-a_L)p_0 + b_L p_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_L \frac{d}{d+1} + (1-b_L)\frac{1}{d+1} \\ (1-a_L)\frac{d}{d+1} + b_L \frac{1}{d+1} \end{pmatrix}$$

$$\Rightarrow \frac{1}{d} p_{L,0} + d\, p_{L,1} = \frac{1}{d+1}\left(a_L + \frac{1-b_L}{d}\right) + \frac{1}{d+1}\left(d^2(1-a_L) + d b_L\right)$$

• In the limit $L \to \infty$: $\lim_{L\to\infty} T^L = \frac{1}{1-c}\begin{pmatrix} 1-b & 1-b \\ 1-a & 1-a \end{pmatrix}$

$\vec{p}_L = \underbrace{T^L}_{} \vec{p}_0$

(*) Markov chain is ergodic

$$\Rightarrow \lim_{L\to\infty} \vec{p}_L = \frac{1}{1-c}\begin{pmatrix} 1-b & 1-b \\ 1-a & 1-a \end{pmatrix}\begin{pmatrix} p_0 \\ p_1 \end{pmatrix} = \frac{1}{1-c}\begin{pmatrix} 1-b \\ 1-a \end{pmatrix}$$

(independent of the initial distribution!).

$\Rightarrow$ purity is $\frac{1}{1-c}\left(\frac{1}{d}(1-b) + d(1-a)\right)$, $c = -1+a+b$

$$a = \frac{d^2}{d^2-1}\left(1 - \frac{\mu(\mathcal{N})}{d}\right) \qquad \mu(\mathcal{N}) = \mathrm{Tr}\left[\left(\mathcal{N}\left(\frac{\mathbb{1}}{d}\right)\right)^2\right]$$

$$b = \frac{d}{d^2-1}\left(d\,\eta(\mathcal{N}) - \frac{1}{d}\right) \qquad \eta(\mathcal{N}) = \frac{1}{d^2}\mathrm{Tr}\left[\left((\mathrm{id}\otimes\mathcal{N})(F)\right)^2\right]$$

If $\mathcal{N}$ is $\overbrace{\text{unital}}^{\mathcal{N}(\mathbb{1}) = \mathbb{1}}$: $\mu(\mathcal{N}) = \mathrm{Tr}\left[\left(\frac{\mathbb{1}}{d}\right)^2\right] = \frac{1}{d} \Rightarrow a = \frac{d^2}{d^2-1}\left(1 - \frac{1}{d^2}\right) = \frac{d^2}{d^2-1}\left(\frac{d^2-1}{d^2}\right) = 1$

$\Rightarrow c = -1 + a + b = b \Rightarrow$ purity is $\frac{1}{1-b}\left(\frac{1}{d}(1-b) + 0\right) = \frac{1}{d} \longrightarrow$ purity of $\frac{\mathbb{1}}{d}$!