What is entanglement?

⊛ Recall that the state vectors of multiple qubits belongs to the tensor-product space $\mathbb{C}^2 \otimes \mathbb{C}^2$.

(Density operators / mixed states belong to $L(\mathbb{C}^2 \otimes \mathbb{C}^2)$.)

$:= \{ M : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2 : M \text{ is linear} \}$

(i.e., 4×4 matrices).

$\hookrightarrow D(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) := \{ \rho \in L(\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}) : \rho = \rho^\dagger, \rho \geq 0, \mathrm{Tr}[\rho] = 1 \}$.

① – A **product state (vector)** has the form:

(State vector) $|\psi\rangle_{AB} = |\phi_1\rangle_A \otimes |\phi_2\rangle_B$, $|\phi_1\rangle \in \mathbb{C}^{d_1}$, $|\phi_2\rangle \in \mathbb{C}^{d_2}$.

(Density operator). $\rho_{AB} = \sigma_A \otimes \tau_B$, $\sigma \in D(\mathbb{C}^{d_1})$, $\omega \in D(\mathbb{C}^{d_2})$.

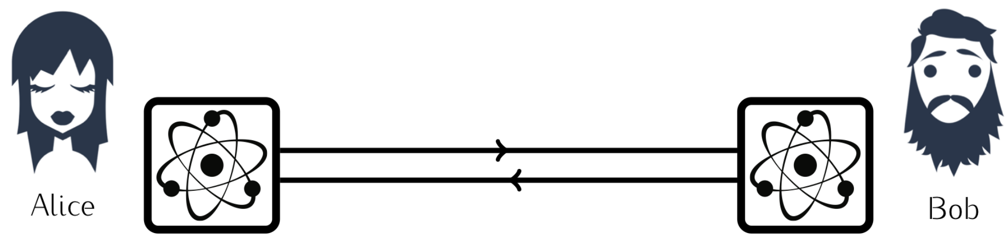
$$\rho_{AB} = \sigma_A \otimes \tau_B$$

Product state

Alice and Bob individually prepare their systems.

- A **Separable state** has the form: $\rho_{AB} = \sum_x p(x)\, \sigma_A^x \otimes \tau_B^x$

↳ Also called "classically correlated" ↳ probabilities.



$$\rho_{AB} = \sum_{x \in \mathcal{X}} p(x)\, \sigma_A^x \otimes \tau_B^x$$

Separable state

Alice and Bob individually prepare their systems via local operations and classical communication.

$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example: $\rho_{AB} = \frac{1}{2}\left(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B\right)$ → $\mathcal{X} = \{0,1\},\ p(0) = p(1) = \frac{1}{2}$

Aside: This state is diagonal in the tensor-product basis

$$\{|0\rangle_A \otimes |0\rangle_B,\ |0\rangle_A \otimes |1\rangle_B,\ |1\rangle_A \otimes |0\rangle_B,\ |1\rangle_A \otimes |1\rangle_B\}$$

$$\rho_{AB} = \frac{1}{2} \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \\ \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \end{array}$$

$\alpha|0\rangle + \beta|1\rangle$

All states of two probabilistic bits can be expressed in this way!

$$\rho_{AB} = \begin{array}{c} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & p_3 & \\ & & & p_4 \end{pmatrix} \\ \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \end{array}$$

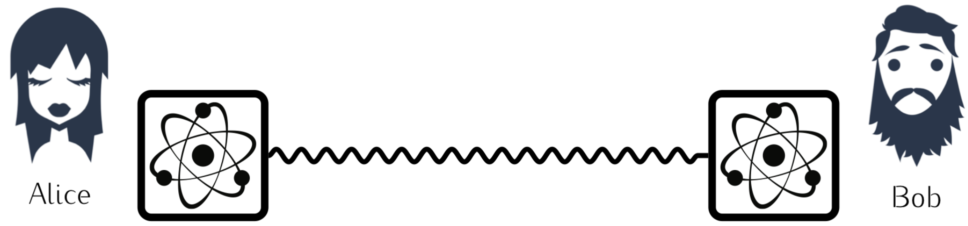
(✱ Even one bit!

$\rho = \begin{pmatrix} p_1 & \\ & p_2 \end{pmatrix} = \frac{1}{2}\left(p_1 |0\rangle\langle 0| + p_2 |1\rangle\langle 1|\right)$

describes an arbitrary state of one bit!)

✱ Any system whose states are always diagonal in a fixed basis is referred to as a "classical system".

- An **entangled state** is NOT a separable state.



$$\rho_{AB} \neq \sum_{x\in\mathcal{X}} p(x)\,\sigma_A^x \otimes \tau_B^x$$

Entangled state

Correlations between Alice and Bob are non-local.
State of the individual systems not sufficient to describe the pair.

② Examples:

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(|+,+\rangle + |-,-\rangle\right)$$

1. The **Bell states**

$$\Phi^\pm = |\Phi^\pm\rangle\langle\Phi^\pm|, \quad |\Phi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|0,0\rangle \pm |1,1\rangle\right)$$

$$\Psi^\pm = |\Psi^\pm\rangle\langle\Psi^\pm|, \quad |\Psi^\pm\rangle = \frac{1}{\sqrt{2}}\left(|0,1\rangle \pm |1,0\rangle\right).$$

Observe that $\{|\Phi^\pm\rangle, |\Psi^\pm\rangle\} \longleftrightarrow \left\{ \underbrace{(Z^z X^x \otimes \mathbb{1})|\Phi^+\rangle}_{|\Phi_{z,x}\rangle} : z,x \in \{0,1\} \right\}$

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Z^0 = \mathbb{1},\ Z^1 = Z, \quad X^0 = \mathbb{1},\ X^1 = X$$

$0,0 \leftrightarrow \Phi^+$

$0,1 \leftrightarrow \Psi^+$

$1,0 \leftrightarrow \Phi^-$

$1,1 \leftrightarrow \Psi^-$

→ In d dimensions: $|\Phi_d^+\rangle := \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k,k\rangle$

(See later for the other d-dimensional Bell states.)

$\circledast$ $|\psi_1\rangle = \alpha_1|0\rangle + \beta_1|1\rangle$, $|\psi_2\rangle = \alpha_2|0\rangle + \beta_2|1\rangle$

$$|\psi_1\rangle \otimes |\psi_2\rangle = \alpha_1\alpha_2|0,0\rangle + \alpha_1\beta_2|0,1\rangle + \beta_1\alpha_2|1,0\rangle + \beta_1\beta_2|1,1\rangle$$

There does not exist values of $\alpha_1, \alpha_2, \beta_1, \beta_2$ such that $|\psi_1\rangle \otimes |\psi_2\rangle = |\Phi^+\rangle$.

$\circledast$ $\{|\Phi_{z,x}\rangle : z,x \in \{0,1\}\}$ is an ONB for $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$\underline{\underline{U}}|z,x\rangle = |\Phi_{z,x}\rangle$$

$\hookrightarrow$ U is a unitary matrix : $U^\dagger U = \mathbb{1} = UU^\dagger \Leftrightarrow U^\dagger = U^{-1}$

(change-of-basis matrix).

$\Rightarrow \{|\Phi_{z,x}\rangle\langle\Phi_{z',x'}| : z,x,z',x' \in \{0,1\}\}$ is an ONB for $L(\mathbb{C}^2 \otimes \mathbb{C}^2)$

$$\Rightarrow \rho_{AB} = \sum_{\substack{z,x \in \{0,1\} \\ z',x' \in \{0,1\}}} \underbrace{C_{z,x}^{z',x'}}_{\in \mathbb{C}} |\Phi_{z,x}\rangle\langle\Phi_{z',x'}|_{AB}$$

$\circledast$ $\sum_{z,x \in \{0,1\}} \underbrace{\Phi_{AB}^{z,x}}_{|\Phi_{z,x}\rangle\langle\Phi_{z,x}|} = \mathbb{1}_{AB} \longrightarrow$ Bell states form a measurement (POVM) — see later.

$\hookrightarrow$ Important ingredient in teleportation.

⊛ "Transpose trick": $(X_{A'} \otimes \mathbb{1}_A)|\Phi_d^+\rangle = (\mathbb{1}_{A'} \otimes X_A^T)|\Phi_d^+\rangle$.

Proof: $(X_{A'} \otimes \mathbb{1}_A)|\Phi_d^+\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1} X_{A'}\overset{\mathbb{1}}{\underset{}{}}|k\rangle_{A'} \otimes |k\rangle_A = \frac{1}{\sqrt{d}}\sum_{k,k'=0}^{d-1} |k'\rangle\overbrace{\langle k'|_A X_{A'}|k\rangle}^{=\langle k|X_A^T|k'\rangle} \otimes |k\rangle_A$

$\mathbb{1} = \sum_k |k\rangle\langle k|$

$$= \frac{1}{\sqrt{d}}\sum_{k,k'=0}^{d-1} |k'\rangle_{A'}\langle k|X_A^T|k'\rangle \otimes |k\rangle_A$$

$$= \frac{1}{\sqrt{d}}\sum_{k,k'=0}^{d-1} |k'\rangle_{A'} \otimes |k\rangle\langle k|_A X_A^T|k'\rangle$$

$$= \frac{1}{\sqrt{d}}\sum_{k'=0}^{d-1} |k'\rangle_{A'} \otimes \underbrace{\left(\sum_{k=0}^{d-1}|k\rangle\langle k|\right)}_{\mathbb{1}_A} X_A^T|k'\rangle$$

$$= \frac{1}{\sqrt{d}}\sum_{k'=0}^{d-1} |k'\rangle_{A'} \otimes X_A^T|k'\rangle$$

$$= (\mathbb{1}_{A'} \otimes X_A^T)\underbrace{\left(\frac{1}{\sqrt{d}}\sum_{k'=0}^{d-1}|k',k'\rangle_{A'A}\right)}_{|\Phi_d^+\rangle}. \quad \blacksquare$$

$$(U \otimes \bar{U})|\Phi^+\rangle = (U \otimes \mathbb{1})\underbrace{(\mathbb{1} \otimes \bar{U})|\Phi^+\rangle}_{\hookrightarrow\ (\bar{U}^T \otimes \mathbb{1})|\Phi^+\rangle = (U^\dagger \otimes \mathbb{1})|\Phi^+\rangle} = (\overbrace{UU^\dagger}^{\mathbb{1}} \otimes \mathbb{1})|\Phi^+\rangle = |\Phi^+\rangle$$

2. Convex mixtures of Bell states might be entangled:

$$\rho_{AB} = \sum_{z,x=0}^{1} \underbrace{p_{z,x}}_{\text{probabilities}} \Phi_{AB}^{z,x}.$$

$(U\otimes\bar{U})\rho_{AB}^{(\lambda)}(U\otimes\bar{U})^\dagger = \rho_{AB}^{(\lambda)}$

→ Isotropic states: $\rho_{AB}^{(\lambda)} = (1-\lambda)\Phi_{0,0} + \frac{\lambda}{3}(\Phi_{0,1} + \Phi_{1,0} + \Phi_{1,1})$

$p_{0,0} = 1-\lambda, \quad p_{0,1} = p_{1,1} = p_{1,1} = \frac{\lambda}{3}, \quad \lambda \in [0,1]$

For $\lambda = 3/4$, $\rho_{AB}^{(3/4)} = \frac{1}{4}(\Phi_{0,0} + \Phi_{0,1} + \Phi_{1,0} + \Phi_{1,1}) = \underbrace{\frac{1}{4}\mathbb{1}_A \otimes \mathbb{1}_B}_{\frac{\mathbb{1}_A}{2}\otimes\frac{\mathbb{1}_B}{2}}$

$\Phi_{1,1} \to |\Psi^-\rangle\langle\Psi^-|$

$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$

→ Werner states: $\omega_{AB}^{(\lambda)} = (1-\lambda)\underbrace{\Phi_{1,1}}_{\text{"singlet" state}} + \frac{\lambda}{3}(\Phi_{0,0} + \Phi_{0,1} + \Phi_{1,0}), \quad \lambda \in [0,1].$

$(U\otimes U)\omega_{AB}^{(\lambda)}(U\otimes U)^\dagger = \omega_{AB}^{(\lambda)}$

→ $\rho_{AB} = (a+b)\Phi_{0,0} + (a-b)\Phi_{1,0} + c\Phi_{0,1} + c\Phi_{1,1}$

Entangled if $a+b > \frac{1}{2}$. → proof later.

Note: $a+b = \langle\Phi^+|\rho_{AB}|\Phi^+\rangle$ → fidelity

3. Multi-party entangled states:

GHZ state: $|GHZ_n\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n})$.

GOING BEYOND BELL'S THEOREM (1989) (arXiv:0712.0921)

Daniel M. Greenberger[1], Michael A. Horne[2], and Anton Zeilinger[3]
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Why is entanglement special?

arXiv > quant-ph > arXiv:quant-ph/0702225

Quantum Physics

[Submitted on 26 Feb 2007 (v1), last revised 20 Apr 2007 (this version, v2)]

Quantum entanglement

Ryszard Horodecki, Pawel Horodecki, Michal Horodecki, Karol Horodecki

arXiv > quant-ph > arXiv:1806.06107

Quantum Physics

[Submitted on 15 Jun 2018 (v1), last revised 20 Apr 2019 (this version, v3)]

Quantum Resource Theories

Eric Chitambar, Gilad Gour

→ QKD (Quantum key distribution)

→ Teleportation

→ Superdense coding

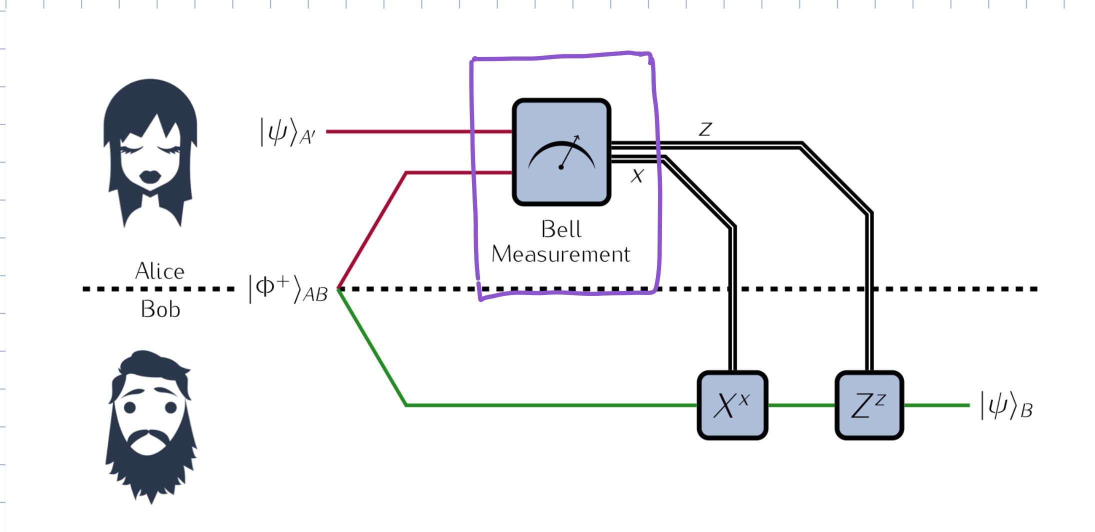
→ Error correction

→ Measurement-based quantum computing.

→ Quantum sensing.

→ Distributed applications (i.e, in a network).

Later in the course!



① Measurements in Quantum mechanics

measurement outcome

$$\rho \longrightarrow \boxed{x} = x$$

$$\{M_x\}_{x \in \mathcal{X}}$$

POVM (positive operator-valued measure):

- $M_x \geq 0$
- $\sum_{x \in \mathcal{X}} M_x = \mathbb{1}$
- $\Pr[x] = \mathrm{Tr}[M_x \rho]$ → Born Rule

Check: $\sum_x \Pr[x] = \sum_x \mathrm{Tr}[M_x \rho] = \mathrm{Tr}\Big[\big(\underbrace{\sum_x M_x}_{\mathbb{1}}\big)\rho\Big] = \mathrm{Tr}[\rho] = 1$

② **Examples** :

1. $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ → "computational basis measurement"
"Pauli-Z meas."

2. $\{\underbrace{|+\rangle\langle +|}, |-\rangle\langle -|\}$ → "Pauli-X meas."

$|+\rangle\underbrace{\langle +|+\rangle}_{=1}\langle +|$
$= |+\rangle\langle +|$

3. $\{|+i\rangle\langle +i|, |-i\rangle\langle -i|\}$ → "Pauli-Y meas."

4. Every ONB on $\mathbb{C}^d$ defines a measurement.

__Proof__: Let $\{|e_i\rangle\langle e_i|\}_{i=1}^{d}$ be an ONB.

$$R := \sum_{i=1}^{d} |e_i\rangle\langle e_i| \quad \hookrightarrow \text{need to prove that } R = \mathbb{1}$$

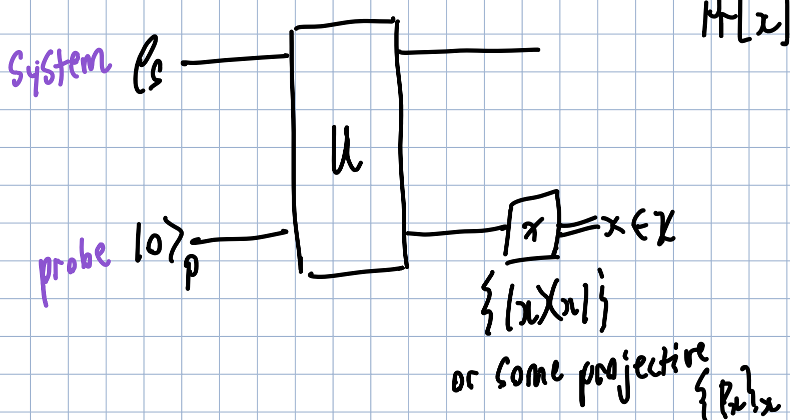
Let $|\psi\rangle \in \mathbb{C}^d$: $R|\psi\rangle = \sum_{i=1}^{d} |e_i\rangle \underbrace{\langle e_i|\psi\rangle}_{\equiv c_i} = |\psi\rangle$ b/c $\{|e_i\rangle\}_{i=1}^{d}$ is ONB.

$\Rightarrow R|\psi\rangle = |\psi\rangle \ \forall\, |\psi\rangle$

$\Rightarrow R = \mathbb{1}$. ∎

⊛ These are all "projective measurements" → $\{P_x\}_{x\in\mathcal{X}}$, $P_x^2 = P_x$
(above ones all have rank one!)

⊛ General non-projective measurement.



$\hookrightarrow \mathrm{Tr}[M_x \rho_s]$

$$\Pr[x] = \mathrm{Tr}\left[(\mathbb{1}_s \otimes P_x)\, U (\rho_s \otimes |0\rangle\langle 0|_p) U^\dagger\right]$$
$$= \mathrm{Tr}\left[U^\dagger (\mathbb{1}_s \otimes P_x) U (\rho_s \otimes |0\rangle\langle 0|_p)\right]$$
$$= \mathrm{Tr}\left[U^\dagger (\mathbb{1}_s \otimes P_x) U (\mathbb{1}_s \otimes |0\rangle_p)(\rho_s)(\mathbb{1}_s \otimes \langle 0|_p)\right]$$

$$= \mathrm{Tr}\Big[\underbrace{(\mathbb{1}_S \otimes \langle 0|_P) U^\dagger (\mathbb{1}_S \otimes P_x) U (\mathbb{1}_S \otimes |0\rangle_P)}_{\equiv M_x} \rho_S\Big]$$

$$\rho_S \otimes |0\rangle\langle 0| \simeq (\mathbb{1}_S \otimes |0\rangle)\rho_S(\mathbb{1}_S \otimes \langle 0|_P)$$

$$\equiv M_x \longrightarrow M_x \geq 0 \checkmark$$

$P \geq 0 \Rightarrow UPU^\dagger \geq 0 \;\forall$ unitaries U.

$$\sum_x M_x = \sum_{x \in \mathcal{X}} (\mathbb{1}_S \otimes \langle 0|_P) U^\dagger (\mathbb{1}_S \otimes P_x) U (\mathbb{1}_S \otimes |0\rangle_P)$$

$$= (\mathbb{1}_S \otimes \langle 0|_P) U^\dagger \Big(\mathbb{1}_S \otimes \underbrace{\big(\sum_x P_x\big)}_{= \mathbb{1}}\Big) U (\mathbb{1}_S \otimes |0\rangle_P)$$

$$= (\mathbb{1}_S \otimes \langle 0|_P)\underbrace{(U^\dagger U)}_{\mathbb{1}_S \otimes \mathbb{1}_P}(\mathbb{1}_S \otimes |0\rangle_P)$$

$$= \mathbb{1}_S \langle 0|0\rangle_P$$

$$= \mathbb{1}_S \checkmark$$

(*) Converse is also true! (Naimark's Theorem).

__Proof__ : Let $\{M_x\}_x$ be a POVM $\rightarrow V := \sum_x |x\rangle \otimes \sqrt{M_x}$

$$\Rightarrow V^\dagger V = \Big(\sum_x \langle x| \otimes \sqrt{M_x}\Big)\Big(\sum_{x'} |x'\rangle \otimes \sqrt{M_{x'}}\Big) = \sum_{x,x'} \langle x|x'\rangle M_x = \sum_x M_x = \mathbb{1}.$$

$\Rightarrow V$ is an isometry.

Also, $$M_x = \underbrace{V^\dagger(|x\rangle\langle x| \otimes \mathbb{1})V}$$

$$\hookrightarrow = \Big(\sum_x \langle x| \otimes \sqrt{M_x}\Big)\big(|x\rangle\langle x| \otimes \mathbb{1}\big)\Big(\sum_{x'} |x'\rangle \otimes \sqrt{M_{x'}}\Big)$$

$$\Rightarrow \mathrm{Tr}[M_x \rho] = \mathrm{Tr}\big[V^\dagger(|x\rangle\langle x| \otimes \mathbb{1})V\rho\big]$$

$$V = U(\mathbb{1} \otimes |0\rangle)$$

$$= \mathrm{Tr}\big[(|x\rangle\langle x| \otimes \mathbb{1})V\rho V^\dagger\big]. \blacksquare$$

5. We want to measure in the $\{|0\rangle, |1\rangle\}$ basis w/ prob. p_1,
$\{|+\rangle, |-\rangle\}$ basis w/ prob. p_2,
$\{|+i\rangle, |-i\rangle\}$ basis w/ prob p_3.

$p_1 + p_2 + p_3 = 1$

$\Rightarrow$ POVM is $\{p_1|0\rangle\langle 0|, p_1|1\rangle\langle 1|, p_2|+\rangle\langle +|, p_2|-\rangle\langle -|, p_3|+i\rangle\langle +i|, p_3|-i\rangle\langle -i|\}$

Check: $p_1|0\rangle\langle 0| + p_1|1\rangle\langle 1| + p_2|+i\rangle\langle +i| + p_2|-\rangle\langle -| + p_3|+i\rangle\langle +i| + p_3|-i\rangle\langle -i|$

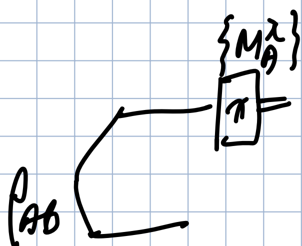
$= p_1\underbrace{(|0\rangle\langle 0| + |1\rangle\langle 1|)}_{\mathbb{1}} + p_2\underbrace{(|+\rangle\langle +| + |-\rangle\langle -|)}_{\mathbb{1}} + p_3\underbrace{(|+i\rangle\langle +i| + |-i\rangle\langle -i|)}_{\mathbb{1}}$

$= \underbrace{(p_1 + p_2 + p_3)}_{=1}\mathbb{1}$

$= \mathbb{1}$ ✓

6. Bell measurement: $\{\Phi^{zx} : z, x \in \{0,1\}\}$

③ "Partial" measurement: measuring only one part of a state.

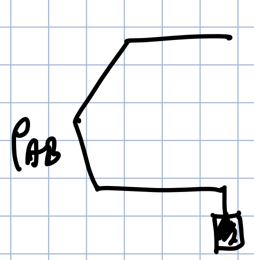


$\Pr[x] = \mathrm{Tr}[(M_A^x \otimes \mathbb{1}_B)\rho_{AB}] = \mathrm{Tr}[M_A^x \rho_A]$

Partial trace of ρ_{AB}.

Also called a "marginal" of ρ_{AB}

$\mathrm{Tr}_A[\rho_{AB}] = \rho_B$.



State from Alice's perspective.

⊛ **Formally** : $\mathrm{Tr}_A\left[X_{AB}\right] = \sum_{k=0}^{d_A-1} \left(\langle k|_A \otimes \mathbb{1}_B\right) X_{AB} \left(|k\rangle_A \otimes \mathbb{1}_B\right)$

(Recall full trace: $\mathrm{Tr}\left[X_{AB}\right] = \sum_{k=0}^{d_A-1} \sum_{l=0}^{d_B-1} \langle k,l|_{AB}\, X_{AB}\, |k,l\rangle_{AB}$.)

$$\mathrm{Tr}_A\left[X_A \otimes Y_B\right] = \mathrm{Tr}\left[X_A\right] Y_B, \quad \mathrm{Tr}_B\left[X_A \otimes Y_B\right] = X_A \mathrm{Tr}\left[Y_B\right]$$

Lemma 1: $\langle \Phi^+|_{A'A} \left(\mathbb{1}_{A'} \otimes M_A\right) |\Phi^+\rangle_{A'A} = \frac{1}{d}\mathrm{Tr}\left[M_A\right]$

Proof:

$$\left(\frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \langle k,k|_{A'A}\right)\left(\mathbb{1}_{A'} \otimes M_A\right)\left(\frac{1}{\sqrt{d}} \sum_{k'=0}^{d-1} |k',k'\rangle_{A'A}\right)$$

$$= \frac{1}{d} \sum_{k,k'=0}^{d-1} \langle k,k|_{A'A} \left(\mathbb{1}_{A'} \otimes M_A\right) |k',k'\rangle_{A'A}$$

$$= \frac{1}{d} \sum_{k,k'=0}^{d-1} \underbrace{\langle k|k'\rangle}_{\delta_{k,k'}} \langle k|M_A|k'\rangle$$

$$= \frac{1}{d} \sum_{k=0}^{d-1} \langle k|M_A|k\rangle = \frac{1}{d}\mathrm{Tr}\left[M_A\right]. \quad \blacksquare$$

Lemma 2: $\langle \Phi^+|_{A'A} \left(\mathbb{1}_{A'} \otimes M_{AB}\right) |\Phi^+\rangle_{A'A} = \frac{1}{d}\mathrm{Tr}_A\left[M_{AB}\right]$

Proof: $\left(\frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} \langle k,k|_{A'A}\right)\left(\mathbb{1}_{A'} \otimes M_{AB}\right)\left(\frac{1}{\sqrt{d}} \sum_{k'=0}^{d-1} |k',k'\rangle_{A'A}\right)$

$$= \frac{1}{d}\sum_{k,k'=0}^{d-1} \underbrace{\langle k|k'\rangle}_{\delta_{k,k'}} \langle k|_A M_{AB} |k'\rangle_A$$

$\langle k|_A \to (\langle k|_A \otimes \mathbb{1}_B)$, $\quad |k'\rangle_A \to (|k'\rangle_A \otimes \mathbb{1}_B)$

$$= \frac{1}{d}\sum_{k=0}^{d-1} \langle k|_A M_{AB} |k\rangle_A = \frac{1}{d}\mathrm{Tr}_A[M_{AB}]. \blacksquare$$

④ **Measurement Channel**: Store the outcome probabilities in a classical (diagonal) state

For POVM $\{M_x\}_{x\in\mathcal{X}}$, $\mathcal{M}(\rho) = \sum_x \mathrm{Tr}[M_x\rho]\,|x\rangle\langle x|$

$$\begin{pmatrix} \mathrm{Tr}[M_{x_1}\rho] & & \\ & \mathrm{Tr}[M_{x_2}\rho] & \\ & & \ddots \end{pmatrix}$$

For a partial measurement. $\{M_A^x\}_{x\in\mathcal{X}}$

$$\to \mathcal{M}_A(\rho_{AB}) = \sum_{x\in\mathcal{X}} |x\rangle\langle x| \otimes \underbrace{\mathrm{Tr}_A[(M_A^x \otimes \mathbb{1}_B)\rho_{AB}]}_{\equiv \tilde{\rho}_B^x}$$

↳ Not a number! (We only trace out the A system).

↳ $\mathrm{Tr}[\tilde{\rho}_B^x] = \Pr[x]$.

$$= \underbrace{\sum_{x\in\mathcal{X}} |x\rangle\langle x| \otimes \tilde{\rho}_B^x}$$

↳ This is a "**classical-quantum state**"

↳ Can be written as a block-diagonal matrix.

$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & \tilde{\rho}_B^{x_1} & & \\ 1 & & \tilde{\rho}_B^{x_2} & \\ 2 & & & \ddots \\ \vdots & & & \end{array}$$

(Recall rules for tensor/Kronecker product!).