Graph states

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A graph state [1–3] is a multi-qubit quantum state defined using graphs.

Consider a graph $G = (V, E)$, which consists of a set V of vertices and a set E of edges. For the purposes of this example, G is an undirected graph, and E is a set of two-element subsets of V. The graph state $|G\rangle$ is an n-qubit quantum state $|G\rangle_{A_1\cdots A_n}$, with $n = |V|$, that is defined as

$$|G\rangle_{A_1\cdots A_n} \coloneqq \sum_{\vec{\alpha}\in\{0,1\}^n} (-1)^{\frac{1}{2}\vec{\alpha}^{\mathsf{T}} A(G)\vec{\alpha}} |\vec{\alpha}\rangle, \tag{1}$$

where $A(G)$ is the adjacency matrix of G, which is defined as

$$A(G)_{i,j} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \in E, \\ 0 & \text{otherwise}, \end{cases} \tag{2}$$

and $\vec{\alpha}$ is the column vector $(\alpha_1, \dots, \alpha_n)^{\mathsf{T}}$. It is easy to show that

$$|G\rangle_{A_1\cdots A_n} = \mathrm{CZ}(G)(|+\rangle_{A_1} \otimes \cdots \otimes |+\rangle_{A_n}), \tag{3}$$

where $|+\rangle \coloneqq \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and

$$\mathrm{CZ}(G) \coloneqq \bigotimes_{\{v_i,v_j\}\in E} \mathrm{CZ}_{A_i A_j}, \tag{4}$$

with $\mathrm{CZ}_{A_i A_j} \coloneqq |0\rangle\langle 0|_{A_i} \otimes \mathbb{1}_{A_j} + |1\rangle\langle 1|_{A_i} \otimes Z_{A_j}$ being the controlled-Z gate.

Now, let

$$Z^{\vec{x}}_{A_1\cdots A_n} \coloneqq Z^{x_1}_{A_1} \otimes \cdots \otimes Z^{x_n}_{A_n}, \tag{5}$$

and observe that

$$\mathrm{CZ}(G) H^{\otimes n} |\vec{x}\rangle = Z^{\vec{x}} |G\rangle \tag{6}$$

for all $\vec{x} \in \{0,1\}^n$. Letting

$$|G^{\vec{x}}\rangle \coloneqq Z^{\vec{x}} |G\rangle, \tag{7}$$

note that $\left\{|G^{\vec{x}}\rangle\langle G^{\vec{x}}|\right\}_{\vec{x}\in\{0,1\}^n}$ is a POVM, due to the fact that

$$|G^{\vec{x}}\rangle = \mathrm{CZ}(G) H^{\otimes n} |\vec{x}\rangle \tag{8}$$

for all $\vec{x} \in \{0,1\}^n$, which follows from (6) and (7), so that

$$\sum_{\vec{x}\in\{0,1\}^n} |G^{\vec{x}}\rangle\langle G^{\vec{x}}| = \mathrm{CZ}(G) H^{\otimes n} \underbrace{\sum_{\vec{x}\in\{0,1\}^n} |\vec{x}\rangle\langle\vec{x}|}_{\mathbb{1}} H^{\otimes n} \mathrm{CZ}(G)^\dagger = \mathbb{1}. \tag{9}$$

References

[1] H. J. Briegel and R. Raussendorf, "Persistent entanglement in arrays of interacting particles", Physical Review Letters **86**, 910 (2001).

[2] R. Raussendorf and H. J. Briegel, "A one-way quantum computer", Physical Review Letters **86**, 5188 (2001).

[3] H. J. Briegel, "Cluster States", in *Compendium of Quantum Physics*, edited by D. Greenberger, K. Hentschel, and F. Weinert (Springer Berlin Heidelberg, Berlin, Heidelberg, 2009), pp. 96–105.