Practical figures of merit and thresholds for entanglement distribution in quantum networks

Sumeet Khatri, Corey Matyas, Aliza Siddiqui, Jonathan Dowling

Department of Physics & Astronomy, Louisiana State University

Phys. Rev. Research 1 (2), 023032

SQuInT 2020

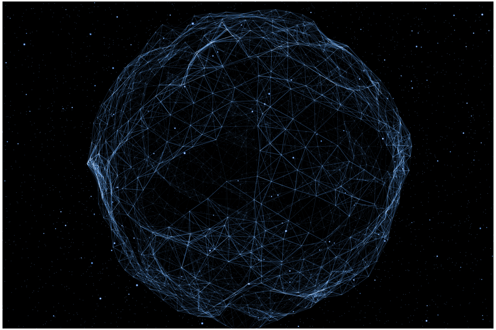
February 9, 2020

Vision: The Quantum Internet

A global interconnected network of quantum networks in which all parties can perform quantum information processing and quantum computing tasks.

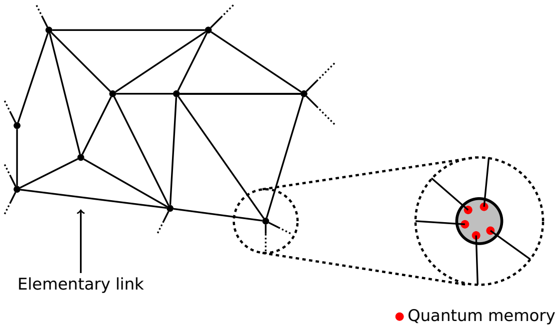
Examples

- Quantum teleportation
- Quantum key distribution
- Distributed quantum computing
- Quantum clock synchronization
- and many others...



Quantum communication networks

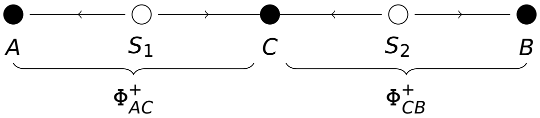
- Nodes are sending/receiving stations.
- Nodes contain at least $d(v)$ quantum memories with coherence time $t^\star$.
- Elementary links represent:
  - Entanglement ⇒ undirected graph;
  - Quantum channels ⇒ directed graph.
- Link creation is probabilistic.
- The graph is *dynamic*: links disappear and reappear over time.



Quantum repeaters

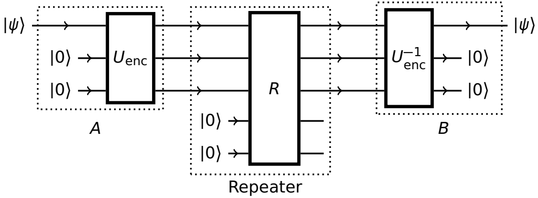
Entanglement-based

- Generate entanglement between desired nodes, then teleport.
- Repeaters perform entanglement swapping and purification.
- Requires two-way communication for heralding.



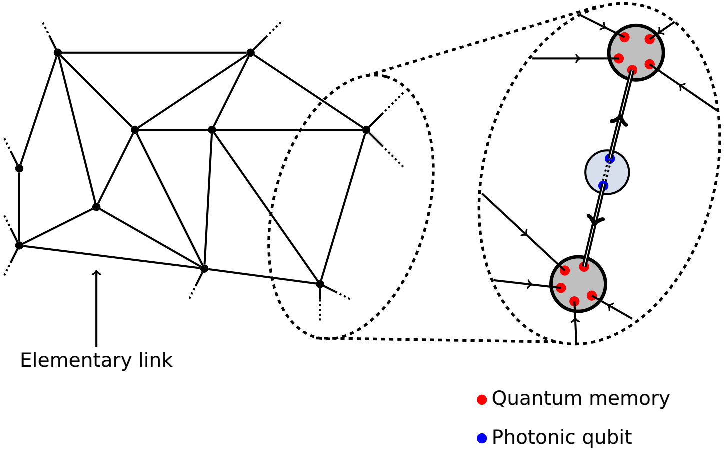
Quantum error correction-based

- Send information directly in encoded form.
- Repeaters apply recovery operations.
- One-way communication, but need fault-tolerant gate operations.



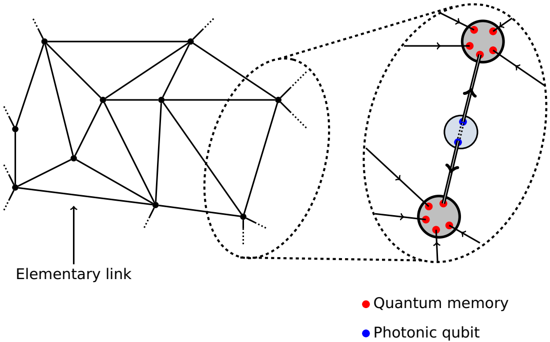
[Sci. Rep. 6, 20463 (2016)]

Entanglement distribution network



★ Our focus: establishing elementary entanglement links.

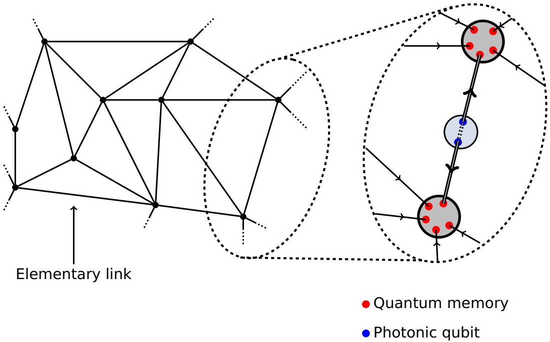
Entanglement distribution model



Sources of loss:

1. Probabilistic sources;
2. Photon transmission through fiber or free space: $\eta = e^{-\alpha \ell}$;
3. Quantum memory read/write inefficiency;
4. Photon detector inefficiency;
5. Probabilistic entanglement purification.

Entanglement distribution model



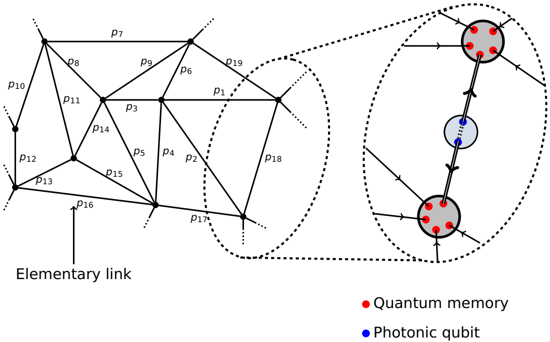
Sources of loss:

1. Probabilistic sources;
2. Photon transmission through fiber or free space: $\eta = e^{-\alpha\ell}$;
3. Quantum memory read/write inefficiency;
4. Photon detector inefficiency;
5. Probabilistic entanglement purification.

★ Every elementary link has success probability p_i, $1 \leq i \leq M$, composed of all loss elements.

★ Every quantum memory has cutoff time $t^\star \Rightarrow$ cutoff number of trials $n^\star = \lfloor Rt^\star \rfloor$.

Entanglement distribution model



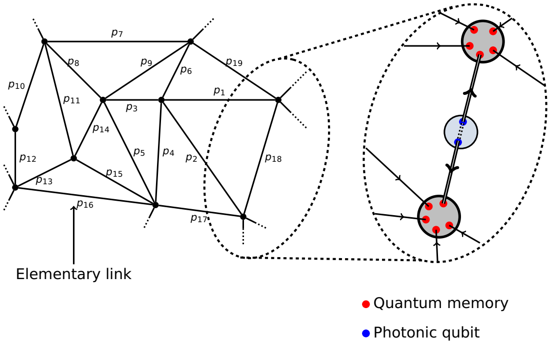
Sources of loss:

1. Probabilistic sources;
2. Photon transmission through fiber or free space: $\eta = e^{-\alpha \ell}$;
3. Quantum memory read/write inefficiency;
4. Photon detector inefficiency;
5. Probabilistic entanglement purification.

★ Every elementary link has success probability p_i, $1 \leq i \leq M$, composed of all loss elements.

★ Every quantum memory has cutoff time $t^\star \Rightarrow$ cutoff number of trials $n^\star = \lfloor R t^\star \rfloor$.

Entanglement distribution model



Sources of loss:

1. Probabilistic sources;
2. Photon transmission through fiber or free space: $\eta = e^{-\alpha \ell}$;
3. Quantum memory read/write inefficiency;
4. Photon detector inefficiency;
5. Probabilistic entanglement purification.

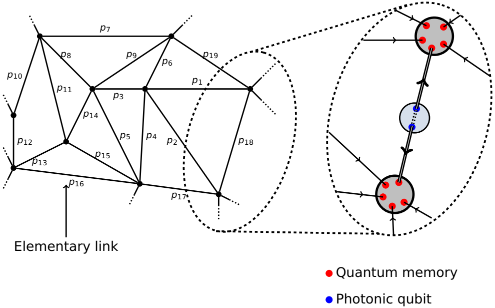
★ Every elementary link has success probability p_i, $1 \le i \le M$, composed of all loss elements.

★ Every quantum memory has cutoff time $t^\star$ ⇒ cutoff number of trials $n^\star = \lfloor R t^\star \rfloor$.

★ What values of p_i and $n^\star$ are acceptable for a “good” network?

Figures of merit in our work

Consider a network based on graph G with memory cutoff $n^{\star}$.



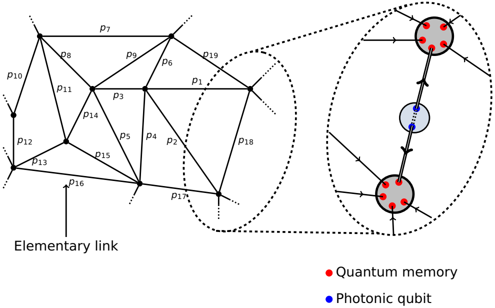
1. How long does it take to establish some number of elementary links in the network?

   $N(M, n^{\star})$: number of time steps needed to establish M elementary links in the network.

   The time is given by $T(M, n^{\star}) = N(M, n^{\star})/R$, where R is the repetition rate (trials per second).

Figures of merit in our work

Consider a network based on graph G with memory cutoff $n^\star$.



2. How many elementary links have been established after a given amount of time?

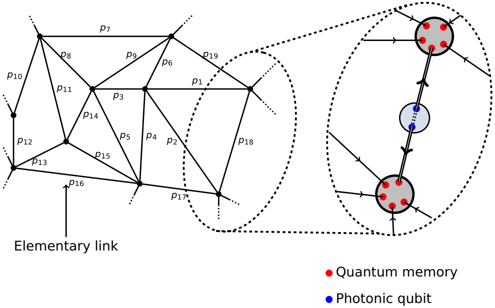
$L_n(G, n^\star)$: number of established elementary links after n time steps.

Can be used to get the *average degree* of the network after n time steps:

$$\mathbb{E}[D_n(G)] = \frac{2\mathbb{E}[L_n(G, n^\star)]}{|G|}.$$

Figures of merit in our work

Consider a network based on graph G with memory cutoff $n^\star$.



3. How big is the largest connected cluster after a given amount of time?

   $S_n^{\max}(G, n^\star)$: size of the largest cluster after n time steps.

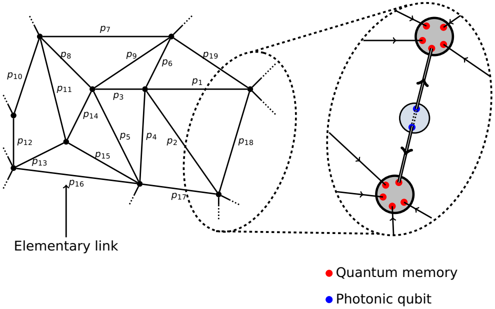
Other graph-theoretic parameters:

- Average clustering coefficient.
- Average distance between two randomly chosen nodes.

(See Barabási, *Network Science*.)

Figures of merit in our work

Consider a network based on graph G with memory cutoff $n^{\star}$.



$\mathbb{E}[N(M, n^{\star})]$: average number of time steps to create M elementary links.

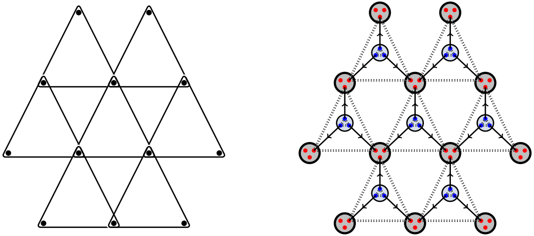
$\mathbb{E}[L_n(G, n^{\star})]$: average number of elementary links after n time steps.

$\mathbb{E}[S_n^{\max}(G, n^{\star})]$: average size of the largest connected cluster after n time steps.

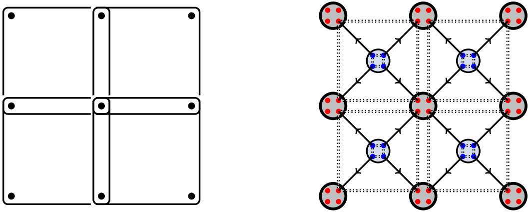
Multipartite elementary links

★ Elementary links can involve multipartite entanglement as well. (Or broadcast channels.)

★ The network then corresponds to a hypergraph.



Each elementary link is a four-partite state.

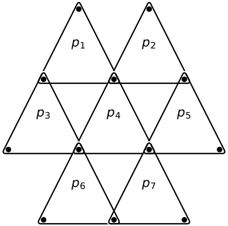


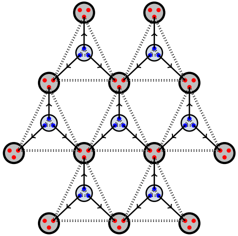
Each elementary link is a four-partite state.

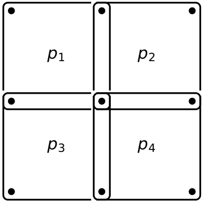
Multipartite elementary links

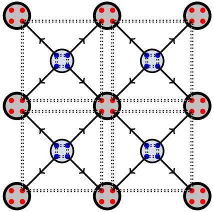
★ Elementary links can involve multipartite entanglement as well. (Or broadcast channels.)

★ The network then corresponds to a hypergraph.









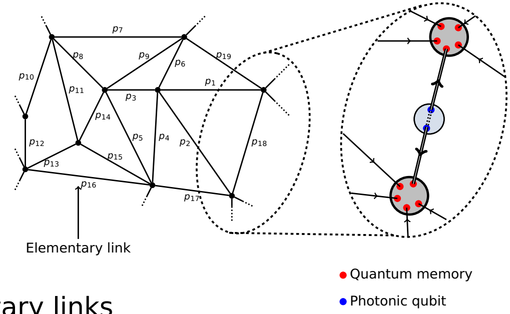
Each elementary link is a four-partite state.

Each elementary link is a four-partite state.

Average link creation time

N_i: number of time steps needed to establish i^{th} elementary link

$$\Pr[N_i = n] = p_i(1-p_i)^{n-1} \Rightarrow \mathbb{E}[N_i] = \frac{1}{p_i}$$



★ All elementary link attempts independent.

★ For simplicity, assume $p_i = p$ for all elementary links
$\Rightarrow N(M, n^\star)$ does not depend on the structure of the graph.

► With $n^\star = 0$ (no quantum memories),

$$\Pr\left[N(M,0) = n\right] = p^M\left(1-p^M\right)^{n-1} \Rightarrow \mathbb{E}\left[N(M,0)\right] = \frac{1}{p^M}$$

Average link creation time

- With $n^\star = \infty$, $N(M,\infty) = \max\{N_1, N_2, \ldots, N_M\}$.

$$\mathbb{E}[N(M,\infty)] = \sum_{k=1}^{M} \binom{M}{k} \frac{(-1)^{k+1}}{1-(1-p)^k}.$$

See also [PRA 83, 012323 (2011)].

- Closed-form expression for $\mathbb{E}[N(M,n^\star)]$ for arbitrary $n^\star$:

$$\mathbb{E}[N(M,n^\star)] = \frac{1-(1-q^{n^\star})^M + (1-q^M)\left[n^\star - \sum_{k=1}^{n^\star-1}(1-q^k)^M\right]}{(1-q^{n^\star+1})^M - q^M(1-q^{n^\star})^M},$$

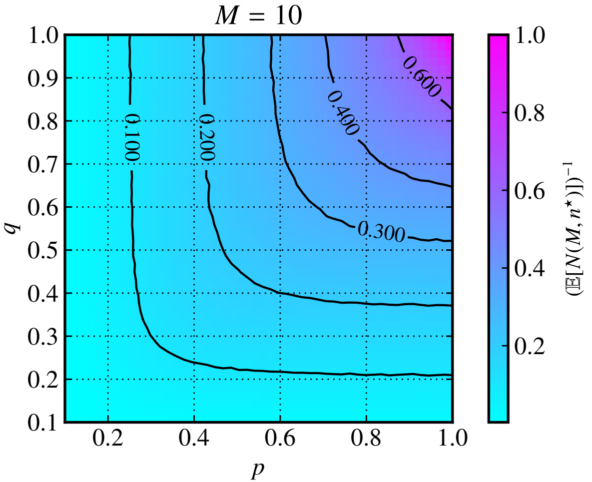
where $q = 1-p$. See [PRA 100, 032322 (2019)], [arXiv:1309.3407].

★ Note that $\mathbb{E}[N(M,\infty)] \leq \mathbb{E}[N(M,n^\star)] \leq \mathbb{E}[N(M,0)]$.

Average link creation time

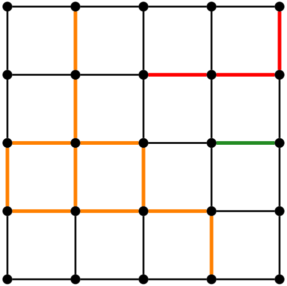
p	M = 10	M = 20
	$n^\star$	$n^\star$
0.01	655	745
0.03	210	250
0.05	125	150
0.1	65	70
0.3	18	20
0.5	9	10

Minimum cutoff $n^\star$ needed to get within 1% of the optimal number $\mathbb{E}[N(M,\infty)]$ of trials.



Network with two different edge probabilities.

Average number of links



$L_n(G, n^\star)$: number of established elementary links after n time steps.

$L^{(j)}(G, n^\star)$: number of established elementary links in the jth time step.

$\left(L^{(j)}(G, n^\star)\right)_{j=1}^{\infty}$ is a Markov chain with memory.

$$L_n(G, n^\star) = \begin{cases} \sum_{j=1}^{n} L^{(j)}(G, n^\star), & n \leq n^\star + 1, \\ \sum_{j=n-n^\star}^{n} L^{(j)}(G, n^\star), & n > n^\star + 1. \end{cases}$$

Average number of links

★ For simplicity, assume $p_i = p$ for all $1 \leq i \leq M$. ($M \equiv$ number of edges in G.)

- For $n^\star = 0$: $L_n(M, 0)$ is a binomial RV

$$\Rightarrow \Pr\left[L_n(G,0) = x\right] = \binom{M}{x} p^x (1-p)^{M-x} \Rightarrow \mathbb{E}\left[L_n(G,0)\right] = Mp.$$

- For $n^\star = \infty$: if x_1 links established in 1st trial, then $M - x_1$ links available for next trial, etc.

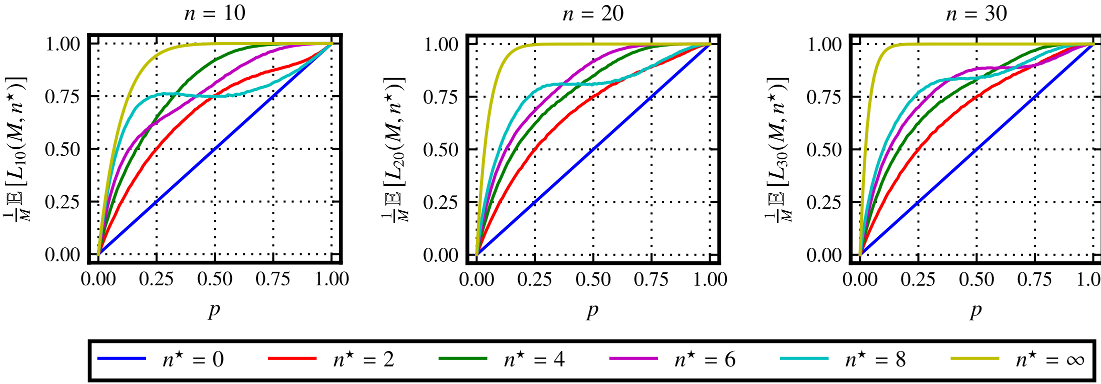
$$\Rightarrow \mathbb{E}\left[L_n(G,\infty)\right] = M(1-(1-p)^n).$$

  More generally,

$$\mathbb{E}[L_n(G,n^\star)] = M(1-(1-p)^n), \quad n^\star \geq n-1.$$

Average number of links

$$p \le \frac{1}{M}\mathbb{E}\left[L_n(G, n^\star)\right] \le 1-(1-p)^n,$$



Average number of links

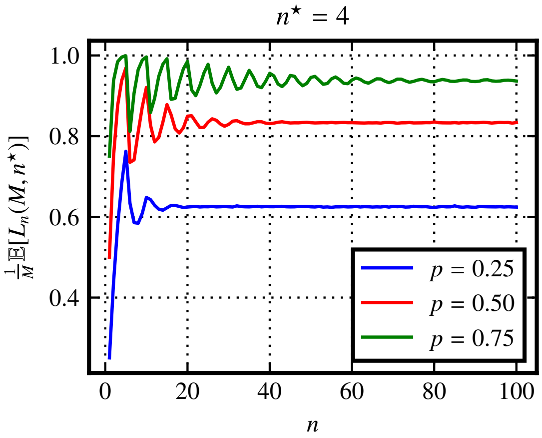
Look at the long-term (steady-state) behavior for fixed $n^\star$ and p: $\lim_{n\to\infty} \frac{1}{M}\mathbb{E}[L_n(G, n^\star)]$.

For $n^\star = 1$: $\lim_{n\to\infty} \frac{1}{M}\mathbb{E}[L_n(G,1)] = \frac{2p}{1+p}$.

For $n^\star = \infty$: $\lim_{n\to\infty} \frac{1}{M}\mathbb{E}[L_n(G,\infty)] = 1$.

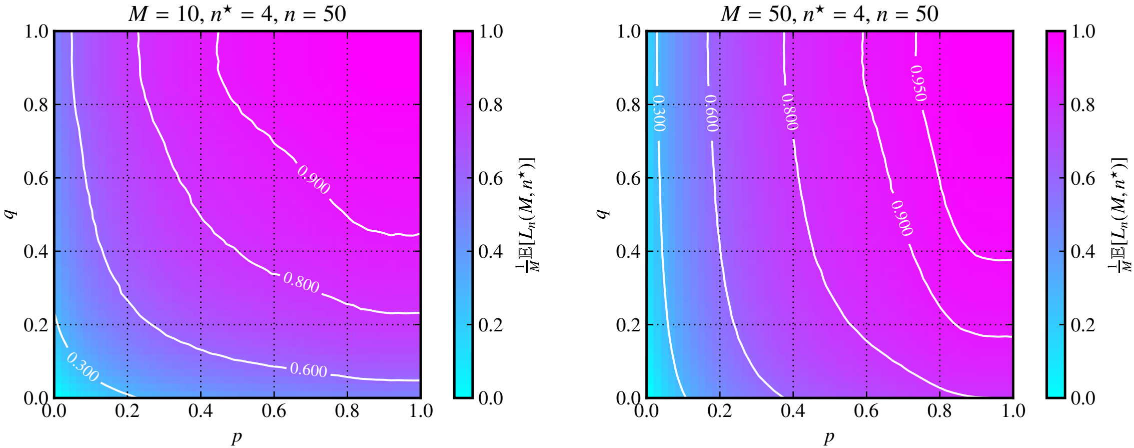
★ Use this to determine required p for a desired fraction f of elementary links in the long-term. For example,

$$n^\star = 1 \quad \Rightarrow \quad p \geq \frac{f}{2-f}.$$



Average number of links

★ Links with two different probabilities.



Even split between p and q.

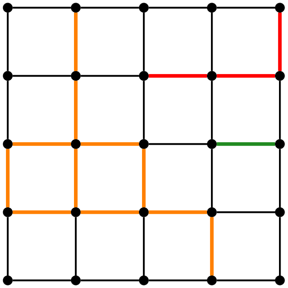
Uneven split between p and q.

Average largest cluster size

$S_n^{\max}(G, n^\star)$: size of the largest cluster after n trials.

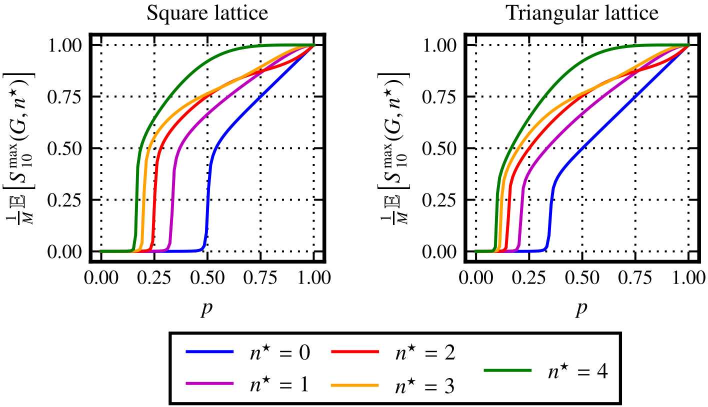
$$S_n^{\max}(G, n^\star) \leq L_n(G, n^\star)$$

$$\boxed{\frac{1}{M}\mathbb{E}\left[S_n^{\max}(G, n^\star)\right] \leq 1-(1-p)^n.}$$



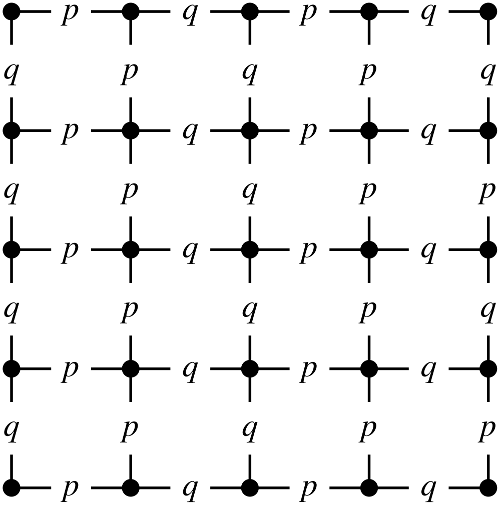
Average largest cluster size

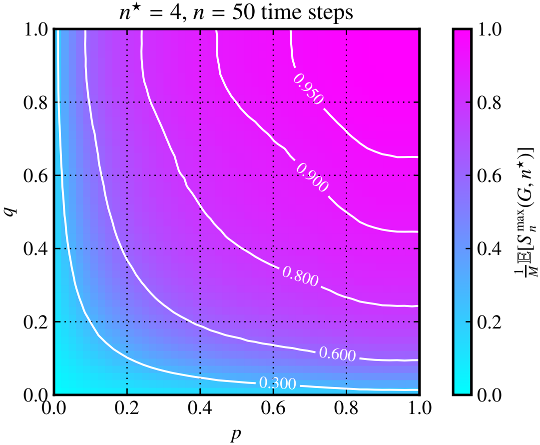
- $n^\star = 0 \Rightarrow$ bond percolation.
- The critical probability quantifies the robustness of the network to loss and other probabilistic aspects. (see also [PRA 97, 012335 (2018)]).



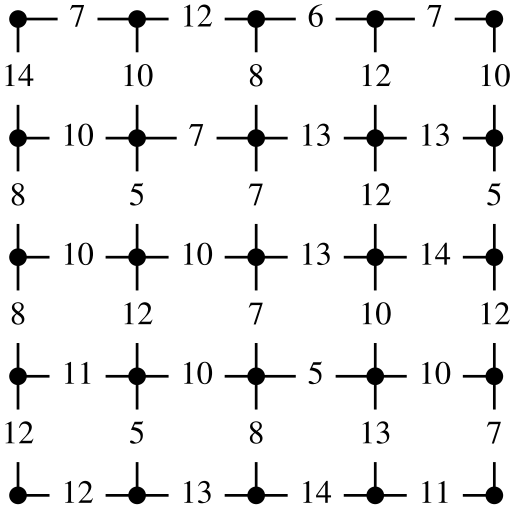
$n^\star$	0	1	2	3	4
Square	0.500	0.336	0.250	0.203	0.166
Triangular	0.347	0.213	0.151	0.117	0.098

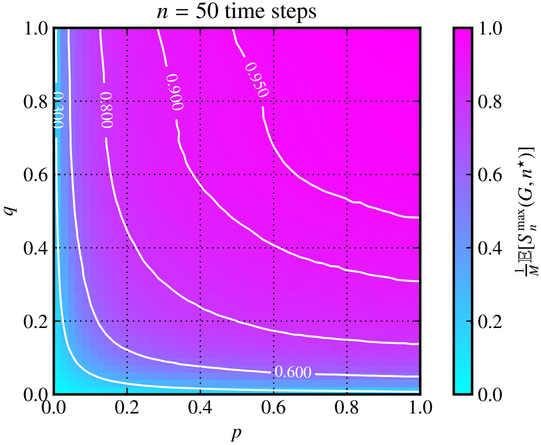
Average largest cluster size





Average largest cluster size





Summary & outlook

We considered practical figures of merit for quantum networks:

1. Average link creation time;
2. Average number of links;
3. Average largest cluster size.

Practical aspects:

- Probabilistic link creation.
- Finite coherence time quantum memories.

Future directions: add fidelity constraints to the figures of merit as a function of $n^\star$. See, e.g., [PRA 97, 062333 (2018)].

Paper available at arXiv:1905.06881.