Pauli Channels

Sumeet Khatri

March 3, 2020

Abstract

We define and state some basic properties of n-qubit Pauli channels.

Table of Contents

1 Introduction

In this note, we consider n-qubit Pauli channels, which are quantum channels that randomly apply an n-qubit Pauli operator to the input state. We start by defining the n-qubit Pauli operators and the corresponding n-qubit Pauli channels. Then, we look at specific examples of n-qubit Pauli channels, such as the depolarizing channel, the dephasing channel, and the bit-flip channel.

2 n-qubit of Pauli operators

The single-qubit Pauli operators are defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} = -\mathrm{i}\sigma_x\sigma_z. \tag{1}$$

For n qubits, we let

$$X^{\vec{j}} := \sigma_x^{j_1} \otimes \sigma_x^{j_2} \otimes \cdots \otimes \sigma_x^{j_n}, \tag{2}$$

be the n-qubit Pauli-X operator, where $\vec{j} = (j_1, j_2, \dots, j_n)$, $j_1, j_2, \dots, j_n \in \{0, 1\}$. Similarly, we let

$$Z^{\vec{k}} := \sigma_z^{k_1} \otimes \sigma_z^{k_2} \otimes \cdots \otimes \sigma_z^{k_n}, \tag{3}$$

be the n-qubit Pauli-Z operator, where $k_1, k_2, \dots, k_n \in \{0, 1\}$. The action of the n-qubit Pauli-X and Pauli-Z operators on standard basis vectors is as follows:

$$X^{\vec{j}}|\vec{\ell}\rangle = |\vec{j} \oplus \vec{l}\rangle, \quad Z^{\vec{k}}|\vec{\ell}\rangle = (-1)^{\vec{k}\cdot\vec{\ell}}|\vec{\ell}\rangle. \tag{4}$$

We note the following properties of the n-qubit Pauli operators that are easy to verify:

$$X^{\vec{j}_1}X^{\vec{j}_2} = X^{\vec{j}_1 \oplus \vec{j}_2}, \quad Z^{\vec{k}_1}Z^{\vec{k}_2} = Z^{\vec{k}_1 \oplus \vec{k}_2}, \quad X^{\vec{j}}Z^{\vec{k}} = (-1)^{\vec{j}\cdot\vec{k}}Z^{\vec{k}}X^{\vec{j}}, \quad \mathrm{Tr}[X^{\vec{j}}Z^{\vec{k}}] = 2^n \delta_{\vec{j},\vec{0}}, \delta_{\vec{k},\vec{0}}. \tag{5}$$

From the last property, and the fact that σ_x and σ_z are linearly independent, it follows that the set $\{X^{\vec{j}}Z^{\vec{k}} : \vec{j}, \vec{k} \in \{0, 1\}^n\}$ is an orthogonal basis for the space of all linear operators acting on the space of n qubits.

Using the properties stated above, we can prove the following identity.

Lemma 1. *For any operator ρ,*

$$\boxed{\frac{1}{2^{2n}} \sum_{\vec{j},\vec{k}} X^{\vec{j}}Z^{\vec{k}}\rho(X^{\vec{j}}Z^{\vec{k}})^\dagger = \mathrm{Tr}[\rho]\frac{\mathbb{1}}{2^n},} \tag{6}$$

Proof. Indeed, we can expand ρ as

$$\rho = \frac{1}{2^n} \sum_{\vec{j},\vec{k}} x_{\vec{j},\vec{k}} X^{\vec{j}}Z^{\vec{k}}, \quad x_{\vec{j},\vec{k}} = \left\langle X^{\vec{j}}Z^{\vec{k}}, \rho \right\rangle = \mathrm{Tr}[(X^{\vec{j}}Z^{\vec{k}})^\dagger \rho]. \tag{7}$$

Note that $\mathrm{Tr}[\rho] = x_{\vec{0},\vec{0}}$. Then, we have

$$\frac{1}{2^{2n}} \sum_{\vec{j},\vec{k}} X^{\vec{j}}Z^{\vec{k}}\rho(X^{\vec{j}}Z^{\vec{k}})^\dagger = \frac{1}{2^{3n}} \sum_{\substack{\vec{j},\vec{k} \\ \vec{j}',\vec{k}'}} x_{\vec{j}',\vec{k}'} X^{\vec{j}}Z^{\vec{k}}(X^{\vec{j}'}Z^{\vec{k}'})(X^{\vec{j}}Z^{\vec{k}})^\dagger \tag{8}$$

$$= \frac{1}{2^{3n}} \frac{1}{2^{3n}} \sum_{\substack{\vec{j},\vec{k} \\ \vec{j}',\vec{k}'}} x_{\vec{j}',\vec{k}'} (-1)^{\vec{k}\cdot\vec{j}'}(-1)^{\vec{k}'\cdot\vec{j}} X^{\vec{j}'}Z^{\vec{k}'}. \tag{9}$$

Now, it holds that

$$\sum_{\vec{j}} (-1)^{\vec{k}'\cdot\vec{j}} = 2^n \delta_{\vec{k}',\vec{0}}, \quad \sum_{\vec{k}} (-1)^{\vec{k}\cdot\vec{j}'} = 2^n \delta_{\vec{j}',\vec{0}}. \tag{10}$$

Therefore,

$$\frac{1}{2^{2n}} \sum_{\vec{j},\vec{k}} X^{\vec{j}}Z^{\vec{k}}\rho(X^{\vec{j}}Z^{\vec{k}})^\dagger = \frac{1}{2^n} x_{\vec{0},\vec{0}} \mathbb{1} = \mathrm{Tr}[\rho]\frac{\mathbb{1}}{2^n}, \tag{11}$$

as required. ◻

Using Eq. (6), we can prove the following.

Lemma 2. *The projector* $|\Phi^+\rangle\langle\Phi^+|$ *onto the* n*-qubit maximally entangled state*

$$|\Phi^+\rangle = \frac{1}{\sqrt{2^n}} \sum_{\vec{j}\in\{0,1\}^n} |\vec{j},\vec{j}\rangle \tag{12}$$

can be written as

$$\boxed{|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2^{2n}} \sum_{\vec{j},\vec{k}} X^{\vec{j}}Z^{\vec{k}} \otimes X^{\vec{j}}Z^{\vec{k}} = \frac{1}{2^{2n}} \sum_{\vec{j},\vec{k}} Z^{\vec{k}}X^{\vec{j}} \otimes Z^{\vec{k}}X^{\vec{j}}.} \tag{13}$$

Proof. To prove this, we make use of the operator vec, which is defined as

$$\mathrm{vec}(X) = (X \otimes \mathbb{1})|\Gamma\rangle, \quad |\Gamma\rangle = \sum_{\vec{j}\in\{0,1\}^n} |\vec{j},\vec{j}\rangle. \tag{14}$$

The vec operator takes any linear operator X and "vectorizes" it, meaning that

$$X = \sum_{i,j} X_{i,j}|i\rangle\langle j| \longmapsto \sum_{i,j} X_{i,j}|i,j\rangle. \tag{15}$$

From this, we see that

$$|\Gamma\rangle = \mathrm{vec}(\mathbb{1}) \tag{16}$$

We also have the following property:

$$\begin{aligned}
\mathrm{vec}(AXB^{\mathsf{T}}) &= (AXB^{\mathsf{T}} \otimes \mathbb{1})|\Gamma\rangle & (17)\\
&= (AX \otimes B)|\Gamma\rangle & (18)\\
&= (A \otimes B)(X \otimes \mathbb{1})|\Gamma\rangle & (19)\\
&= (A \otimes B)\mathrm{vec}(X) & (20)\\
\Rightarrow \mathrm{vec}(AXB^{\mathsf{T}}) &= (A \otimes B)\mathrm{vec}(X), & (21)
\end{aligned}$$

and we have that

$$\mathrm{Tr}[X] = \langle\Gamma|(X \otimes \mathbb{1})|\Gamma\rangle = \langle\Gamma|\mathrm{vec}(X). \tag{22}$$

Now, taking $\mathrm{vec}(\cdot)$ on both sides of Eq. (6), we obtain

$$\begin{aligned}
\frac{1}{2^{2n}} \sum_{\vec{j},\vec{k}} \mathrm{vec}\left(X^{\vec{j}}Z^{\vec{k}}\rho(X^{\vec{j}}Z^{\vec{k}})^\dagger\right) &= \frac{|\Gamma\rangle\langle\Gamma|\mathrm{vec}(\rho)}{2^n} = |\Phi^+\rangle\langle\Phi^+|\mathrm{vec}(\rho) & (23)\\
\Rightarrow \frac{1}{2^{2n}} \sum_{\vec{j},\vec{k}} (X^{\vec{j}}Z^{\vec{k}} \otimes \overline{X^{\vec{j}}Z^{\vec{k}}})\mathrm{vec}(\rho) &= |\Phi^+\rangle\langle\Phi^+|\mathrm{vec}(\rho) & (24)\\
\Rightarrow \frac{1}{2^{2n}} \sum_{\vec{j},\vec{k}} (X^{\vec{j}}Z^{\vec{k}} \otimes X^{\vec{j}}Z^{\vec{k}})\mathrm{vec}(\rho) &= |\Phi^+\rangle\langle\Phi^+|\mathrm{vec}(\rho) & (25)
\end{aligned}$$

Since the operator ρ is arbitrary, we obtain the desired result. □

3 n-qubit Pauli channels

An n-qubit Pauli channel corresponds to the action of random n-qubit Pauli operators on a quantum state ρ according to some probability distribution. Let $\mathcal{P}^{(n)}$ denote an n-qubit Pauli channel. Then the action of $\mathcal{P}^{(n)}$ on the state ρ is given by

$$\mathcal{P}_{\vec{p}}^{(n)}(\rho) = \sum_{\vec{j},\vec{k}} p_{\vec{l},\vec{k}} X^{\vec{j}} Z^{\vec{k}} \rho (X^{\vec{j}} Z^{\vec{k}})^{\dagger}, \tag{26}$$

where $0 \leq p_{\vec{j},\vec{k}} \leq 1$, and $\sum_{\vec{j},\vec{k}} p_{\vec{l},\vec{k}} = 1$. Using the properties in Eq. (5), we find that

$$\begin{aligned} \mathcal{P}_{\vec{p}}^{(n)}(X^{\vec{a}} Z^{\vec{b}}) &= \sum_{\vec{j},\vec{k}} p_{\vec{j},\vec{k}} X^{\vec{j}} Z^{\vec{k}} X^{\vec{a}} Z^{\vec{b}} Z^{\vec{k}} X^{\vec{j}} &(27)\\ &= \sum_{\vec{j},\vec{k}} (-1)^{\vec{a}\cdot\vec{k}} (-1)^{\vec{b}\cdot\vec{j}} p_{\vec{j},\vec{k}} X^{\vec{a}} Z^{\vec{b}} &(28)\\ &= c_{\vec{a},\vec{b}} X^{\vec{a}} Z^{\vec{b}}, &(29) \end{aligned}$$

where

$$c_{\vec{a},\vec{b}} \coloneqq \sum_{\vec{j},\vec{k}} (-1)^{\vec{a}\cdot\vec{k}} (-1)^{\vec{b}\cdot\vec{j}} p_{\vec{j},\vec{k}}. \tag{30}$$

We have that $-1 \leq c_{\vec{a},\vec{b}} \leq 1$ for all $\vec{a}, \vec{b} \in \{0,1\}^n$.

Observe that Pauli channels are diagonal in the Pauli basis, in the sense that if we define the matrix $P_{\vec{p}}^{(n)}$ as

$$(P_{\vec{p}}^{(n)})_{\substack{\vec{j},\vec{k}\\ \vec{j}',\vec{k}'}} = \frac{1}{2^n} \operatorname{Tr}[(X^{\vec{j}} Z^{\vec{k}})^{\dagger} \mathcal{P}_{\vec{p}}^{(n)} (X^{\vec{j}'} Z^{\vec{k}'})], \tag{31}$$

then by using the properties in (5) we get

$$(P_{\vec{p}}^{(n)})_{\substack{\vec{j},\vec{k}\\ \vec{j}',\vec{k}'}} = c_{\vec{j},\vec{k}} \delta_{\vec{j},\vec{j}'} \delta_{\vec{k},\vec{k}'}. \tag{32}$$

4 Depolarizing channel

Given the general form of an n-qubit Pauli channel, namely

$$\mathcal{P}_{\vec{p}}^{(n)}(\cdot) = \sum_{\vec{j},\vec{k}} p_{\vec{j},\vec{k}} X^{\vec{j}} Z^{\vec{k}} (\cdot) (X^{\vec{j}} Z^{\vec{k}})^{\dagger}, \tag{33}$$

the depolarizing channel is defined by taking

$$p_{\vec{j},\vec{k}} = \begin{cases} 1-p & \text{if } \vec{j} = \vec{k} = \vec{0}, \\ \frac{p}{2^{2n}-1} & \text{otherwise,} \end{cases} \tag{34}$$

for some $p \in [0,1]$. We thus have the following definition of the n-qubit depolarizing channel $\mathcal{D}_n$:

$$\mathcal{D}_p^{(n)}(\rho) = (1-p)\rho + \frac{p}{2^{2n}-1} \sum_{(\vec{j},\vec{k})\neq(\vec{0},\vec{0})} X^{\vec{j}} Z^{\vec{k}} \rho (X^{\vec{j}} Z^{\vec{k}})^\dagger. \tag{35}$$

In order to determine the coefficients $c_{\vec{a},\vec{b}}$ of this channel, let us first write the action of the n-qubit depolarizing in a simpler form by making use of the identity in Eq. (6). We get

$$\mathcal{D}_p^{(n)}(\rho) = \left(1 - \frac{2^{2n}}{2^{2n}-1}p\right)\rho + \frac{2^{2n}}{2^{2n}-1} p \operatorname{Tr}[\rho] \frac{\mathbb{1}}{2^n} \tag{36}$$

for all operators ρ. Now, since the operators $X^{\vec{a}} Z^{\vec{b}}$ are traceless for all $(\vec{a},\vec{b}) \neq (\vec{0},\vec{0})$, we have

$$\mathcal{D}_p^{(n)}(X^{\vec{a}} Z^{\vec{b}}) = \left(1 - \frac{2^{2n}}{2^{2n}-1}p\right) X^{\vec{a}} Z^{\vec{b}} \quad \forall\ (\vec{a},\vec{b}) \neq (\vec{0},\vec{0}) \Rightarrow c_{\vec{a},\vec{b}} = 1 - \frac{2^{2n}}{2^{2n}-1}p, \tag{37}$$

$$\mathcal{D}_p^{(n)}(\mathbb{1}) = \mathbb{1} \Rightarrow c_{\vec{0},\vec{0}} = 1. \tag{38}$$

Therefore, $c_{\vec{a},\vec{b}} \geq 0$ if and only if $p \leq \frac{2^{2n}-1}{2^{2n}}$.

5 Dephasing channels

The dephasing channel on n qubits is defined as follows:

$$\mathcal{Z}_{\vec{q}}^{(n)}(\rho) = \sum_{\vec{k}} q_{\vec{k}} Z^{\vec{k}} \rho Z^{\vec{k}}. \tag{39}$$

Then,

$$\mathcal{Z}_{\vec{q}}^{(n)}(X^{\vec{a}} Z^{\vec{b}}) = \sum_{\vec{k}} q_{\vec{k}} Z^{\vec{k}} (X^{\vec{a}} Z^{\vec{b}}) Z^{\vec{k}} \tag{40}$$

$$= \sum_{\vec{k}} q_{\vec{k}} (-1)^{\vec{a}\cdot\vec{k}} X^{\vec{a}} Z^{\vec{b}}, \tag{41}$$

so that

$$c_{\vec{a},\vec{b}} = \sum_{\vec{k}} q_{\vec{k}} (-1)^{\vec{a}\cdot\vec{k}}. \tag{42}$$

Now, a special case of the n-qubit dephasing channels is the one in which

$$q_{\vec{k}} = \begin{cases} 1-p & \text{if } \vec{k} = \vec{0}, \\ p & \text{if } \vec{k} = \vec{1}, \\ 0 & \text{otherwise,} \end{cases} \tag{43}$$

for some $p \in [0,1]$. In this case,

$$\widetilde{\mathcal{Z}}_p^{(n)}(\rho) = (1-p)\rho + p(Z \otimes \cdots \otimes Z)\rho(Z \otimes \cdots \otimes Z). \tag{44}$$

Then, we have

$$c_{\vec{a},\vec{b}} = q_{\vec{0}} + (-1)^{\vec{a}\cdot\vec{1}} q_{\vec{1}} = 1 - p + (-1)^{|\vec{a}|} p = \begin{cases} 1 & \text{if } |\vec{a}| \text{ even}, \\ 1 - 2p & \text{if } |\vec{a}| \text{ odd}, \end{cases} \tag{45}$$

where $|\vec{a}|$ denotes the Hamming weight (number of ones) in the bit string $\vec{a}$. Therefore, in this case, we have $c_{\vec{a},\vec{b}} \geq 0$ if and only if $p \leq \frac{1}{2}$.

Another special case is the one in which $q_{\vec{k}} = \frac{1}{2^n}$ for all $\vec{k}$, so that

$$\overline{\mathcal{Z}}^{(n)}(\rho) \coloneqq \frac{1}{2^n} \sum_{\vec{k}} Z^{\vec{k}} \rho Z^{\vec{k}}. \tag{46}$$

For any input operator ρ, this channel returns a diagonal state consisting of the diagonal elements of ρ in the standard basis. Indeed, let ρ be written in the standard basis as

$$\rho = \sum_{\vec{k},\vec{\ell}} x_{\vec{k},\vec{\ell}} |\vec{k}\rangle\langle\vec{\ell}|. \tag{47}$$

Then,

$$\begin{aligned}
\overline{\mathcal{Z}}^{(n)}(\rho) &= \frac{1}{2^n} \sum_{\vec{j}} Z^{\vec{j}} \rho Z^{\vec{j}} && (48)\\
&= \frac{1}{2^n} \sum_{\vec{j},\vec{k},\vec{\ell}} x_{\vec{k},\vec{\ell}} Z^{\vec{j}} |\vec{k}\rangle\langle \ell | Z^{\vec{j}} && (49)\\
&= \frac{1}{2^n} \sum_{\vec{j},\vec{k},\vec{\ell}} x_{\vec{k},\vec{\ell}} (-1)^{\vec{j}\cdot(\vec{k}\oplus\vec{\ell})} |\vec{k}\rangle\langle\vec{\ell}| && (50)\\
&= \frac{1}{2^n} \sum_{\vec{k},\vec{\ell}} x_{\vec{k},\vec{\ell}} x_{\vec{k},\vec{\ell}} \underbrace{\left(\sum_{\vec{j}} (-1)^{\vec{j}\cdot(\vec{k}\oplus\vec{\ell})} \right)}_{2^n \delta_{\vec{k},\vec{\ell}}} |\vec{k}\rangle\langle\vec{\ell}| && (51)\\
&= \sum_{\vec{k},\vec{\ell}} x_{\vec{k},\vec{k}} |\vec{k}\rangle\langle\vec{k}|. && (52)
\end{aligned}$$

In this special case, we have that

$$c_{\vec{a},\vec{b}} = \frac{1}{2^n} \sum_{\vec{k}} (-1)^{\vec{a}\cdot\vec{k}} = \delta_{\vec{a},\vec{0}} \tag{53}$$

for all $\vec{a}, \vec{b}$.

The dephasing channel in Eq. (46) can be generalized as follows:

$$\mathcal{Z}_p^{(n)}(\rho) = (1-p)\rho + \frac{p}{2^n - 1} \sum_{\vec{k} \neq \vec{0}} Z^{\vec{k}} \rho Z^{\vec{k}}, \tag{54}$$

where $p \in [0,1]$. Then, $\overline{\mathcal{Z}}^{(n)} = \mathcal{Z}^{(n)}_{1-\frac{1}{2^n}}$. In this case, we have

$$c_{\vec{a},\vec{b}} = 1 - \frac{2^n}{2^n-1}p + \frac{p}{2^n-1}\delta_{\vec{a},\vec{0}} \tag{55}$$

for all $\vec{a}, \vec{b}$. This is non-negative if and only if $p \leq \frac{2^n-1}{2^n}$.

6 Bit-flip channels

The bit-flip channel on n qubits is defined analogously to the n-qubit dephasing channel above:

$$\mathcal{X}^{(n)}_{\vec{r}}(\rho) = \sum_{\vec{k}} r_{\vec{k}} X^{\vec{k}} \rho X^{\vec{k}}. \tag{56}$$

Then, by the same arguments as for the dephasing channel, we get

$$c_{\vec{a},\vec{b}} = \sum_{\vec{k}} r_{\vec{k}} (-1)^{\vec{b}\cdot\vec{k}}. \tag{57}$$

As with the dephasing channel, we define

$$\widetilde{\mathcal{X}}^{(n)}_p(\rho) = (1-p)\rho + p(X \otimes \cdots \otimes X)\rho(X \otimes \cdots \otimes X), \tag{58}$$

so that

$$c_{\vec{a},\vec{b}} = \begin{cases} 1 & \text{if } |\vec{b}| \text{ even,} \\ 1-2p & \text{if } |\vec{b}| \text{ odd.} \end{cases} \tag{59}$$

We can also define the channel

$$\overline{\mathcal{X}}^{(n)} = \frac{1}{2^n} \sum_{\vec{j}} X^{\vec{j}} \rho X^{\vec{j}}, \tag{60}$$

which has the following effect on any operator $\rho = \sum_{\vec{k},\vec{\ell}} x_{\vec{k},\vec{\ell}} |\vec{k}\rangle\langle\vec{\ell}|$:

$$\overline{\mathcal{X}}^{(n)} = \frac{1}{2^n} \sum_{\vec{j}} X^{\vec{j}} \rho X^{\vec{j}} \tag{61}$$

$$= \frac{1}{2^n} \sum_{\vec{j},\vec{k},\vec{\ell}} x_{\vec{k},\vec{\ell}} |\vec{k} \oplus \vec{j}\rangle\langle\vec{\ell} \oplus \vec{j}| \tag{62}$$

$$= \sum_{\vec{k}',\vec{\ell}'} \left(\frac{1}{2^n} \sum_{\vec{j}} x_{\vec{k}'\oplus\vec{j},\vec{\ell}'\oplus\vec{j}} \right) |\vec{k}'\rangle\langle\vec{\ell}'|. \tag{63}$$

In this special case, we have

$$c_{\vec{a},\vec{b}} = \delta_{\vec{b},\vec{0}}. \tag{64}$$

Finally, we can define the channel

$$\mathcal{X}_p^{(n)}(\rho) = (1-p)\rho + \frac{p}{2^n - 1} \sum_{\vec{k} \neq \vec{0}} X^{\vec{k}} \rho X^{\vec{k}}, \tag{65}$$

where $p \in [0,1]$. Then, $\overline{\mathcal{X}}^{(n)} = \mathcal{X}^{(n)}_{1-\frac{1}{2^n}}$. In this case, we have

$$c_{\vec{a},\vec{b}} = 1 - \frac{2^n}{2^n - 1} p + \frac{p}{2^n - 1} \delta_{\vec{b},\vec{0}}. \tag{66}$$