Quantum Channels : In quantum information, q. channels represent:

- Physical evolution of systems. (physics)
- Information transmission b/w spatially distant locations (inf. theory)
- Mathematical tools to help with calculations (math)

- **Schrödinger Equation** : $i\hbar \frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle$ (For pure states).

Hamiltonian.

$$\Rightarrow \frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle$$

- For mixed states: Recall mixed states can be written as

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \Rightarrow \frac{d}{dt}\rho(t) = \sum_i \frac{d}{dt}\left(p_i|\psi_i\rangle\langle\psi_i|\right)$$

$$= \sum_i \left[\cancel{\left(\frac{d}{dt}p_i\right)}|\psi_i\rangle\langle\psi_i| + p_i\left(\underbrace{\frac{d}{dt}|\psi_i\rangle}_{\to -\frac{i}{\hbar}H|\psi_i\rangle}\right)\langle\psi_i| + p_i|\psi_i\rangle\left(\underbrace{\frac{d}{dt}\langle\psi_i|}_{\to \langle\psi_i|H^\dagger\left(\frac{i}{\hbar}\right) = \langle\psi_i|H\left(\frac{i}{\hbar}\right)}\right)\right]$$

$=0$ (assumption)

$$= \sum_i \left[p_i\left(-\frac{i}{\hbar}\right)H|\psi_i\rangle\langle\psi_i| + p_i\left(\frac{i}{\hbar}\right)|\psi_i\rangle\langle\psi_i|H\right]$$

$$= -\frac{i}{\hbar}\left(H\underbrace{\left(\sum_i p_i|\psi_i\rangle\langle\psi_i|\right)}_{\rho} - \underbrace{\left(\sum_i p_i|\psi_i\rangle\langle\psi_i|\right)}_{\rho}H\right)$$

$$= -\frac{i}{\hbar}\left(H\rho - \rho H\right)$$

$$= -\frac{i}{\hbar}[H,\rho] \Rightarrow \boxed{\frac{d}{dt}\rho(t) = -\frac{i}{\hbar}[H(t),\rho(t)]}$$ (von Neumann equation)

– General Solution : Unitary dynamics:

$$\boxed{\rho(t) = U(t_0,t)\,\rho(t_0)\,U(t_0,t)^\dagger}$$

(For H time-indep. : $U(t_0,t) = e^{-\frac{i}{\hbar}(t-t_0)H}$)

– Open Systems : Systems interacting with (inaccessible) environment.

System + environment evolve jointly according to Schrödinger equation:

$\rho_{SE}(t_0) = |\psi\rangle\langle\psi|_S \otimes |0\rangle\langle 0|_E$ → Doesn't matter what State, just has to be pure!

$\rightarrow \rho_{SE}(t) = U(t_0,t)\left(|\psi\rangle\langle\psi|_S \otimes |0\rangle\langle 0|_E\right)U(t_0,t)^\dagger$

Environment inaccessible => trace out environment + get state of System alone.

$$\rho_S(t) = \mathrm{Tr}_E\left[\rho_{SE}(t)\right] = \mathrm{Tr}_E\left[U(t_0,t)\left(|\psi\rangle\langle\psi|_S \otimes |0\rangle\langle 0|_E\right)U(t_0,t)^\dagger\right]$$

$$\equiv \mathcal{N}\left(|\psi\rangle\langle\psi|_S\right)$$

$$U(t_0,t) = \sum_{i,j=0}^{d_E-1} U_{ij} \otimes |i\rangle\langle j|_E$$

$\Rightarrow U^\dagger U = \left(\sum_{i,j=0}^{d-1} U_{ij}^\dagger \otimes |j\rangle\langle i|\right)\left(\sum_{i',j'=0}^{d-1} U_{i'j'} \otimes |i'\rangle\langle j'|\right) = \sum_{i,j,j'=0}^{d-1} U_{ij}^\dagger U_{ij'} \otimes |j\rangle\langle j'|$

$= \underset{SE}{\mathbb{1}} = \sum_{j=0}^{d-1} \mathbb{1}_S \otimes |j\rangle\langle j|_E$

$\Rightarrow \sum_{i=0}^{d-1} U_{ij}^\dagger U_{ij'} = \mathbb{1}\,\delta_{jj'}$

$\forall\ 0 \le j,j' \le d-1$

$$= \mathrm{Tr}_E\left[\left(\sum_{i,j=0}^{d_E-1} U_{ij} \otimes |i\rangle\langle j|_E\right)\left(|\psi\rangle\langle\psi|_S \otimes |0\rangle\langle 0|_E\right)\left(\sum_{i',j'=0}^{d_E-1} U_{i'j'}^\dagger \otimes |j'\rangle\langle i'|_E\right)\right]$$

$$= \sum_{i,j,i',j'=0}^{d_E-1} U_{ij}\,|\psi\rangle\langle\psi|_S\,U_{i'j'}^\dagger \otimes \underbrace{\mathrm{Tr}\left[|i\rangle\langle j|0\rangle\langle 0|j'\rangle\langle i'|\right]}_{\delta_{j,0}\,\delta_{j',0}\,\delta_{i',i}}$$

$$= \sum_{k=0}^{d_E-1} U_{k,0}\,|\psi\rangle\langle\psi|_S\,U_{k,0}^\dagger \qquad \Rightarrow \sum_{k=0}^{d-1} U_{k,0}^\dagger U_{k,0} = \mathbb{1}_S$$

$\Rightarrow$ Two general forms of q. channels:

① $\mathcal{N}(\rho) = \underbrace{\sum_k A_k \rho A_k^\dagger}_{CP}, \ \underbrace{\sum_k A_k^\dagger A_k = \mathbb{1}}_{TP}$ (Kraus).

② $\mathcal{N}(\rho) = \mathrm{Tr}_E\left[U(\rho \otimes |\psi\rangle\langle\psi|_E)U^\dagger\right]$ (Schrödinger).

Mathematically, these maps are completely-positive (CP) and trace preserving (TP).

$\hookrightarrow (\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(X_{RA}) \geq 0 \ \forall \ X_{RA} \geq 0, \ d_R \geq 1$. (entangled states should be mapped to states).

$\hookrightarrow \mathrm{Tr}[\mathcal{N}_{A\to B}(\rho_A)] = \mathrm{Tr}[\rho_A] \ \forall \ \rho_A$ (probability preservation).

③ Choi: $\rho^{\mathcal{N}}_{RB} = (\mathrm{id}_R \otimes \mathcal{N}_{A\to B})(|\Phi^+\rangle\langle\Phi^+|_{RA})$ $\left(|\Phi^+\rangle_{RA} = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle_R |i\rangle_A \right)$

$$= \frac{1}{d} \underbrace{\sum_{i,j=0}^{d-1} |i\rangle\langle j|_R \otimes \mathcal{N}_{A\to B}(|i\rangle\langle j|_A)}_{\Gamma^{\mathcal{N}}_{RB}}$$

$\boxed{\circledast \ \mathcal{N}_{A\to B} \ \text{CP iff } \rho^{\mathcal{N}}_{AB} \geq 0}$

$\boxed{\text{TP iff } \mathrm{Tr}_B[\rho^{\mathcal{N}}_{AB}] = \frac{\mathbb{1}_A}{d}}$

$$\Rightarrow \mathrm{Tr}_A\left[(\rho_A^T \otimes \mathbb{1}_B)\Gamma^{\mathcal{N}}_{AB}\right] = \mathrm{Tr}_A\left[(\rho_A^T \otimes \mathbb{1}_B)\mathcal{N}_{A'\to B}(\Gamma_{AA'})\right]$$
$$= \mathrm{Tr}_A\left[(\mathbb{1}_{AB})\mathcal{N}_{A'\to B}\left(\underbrace{(\rho_A^T \otimes \mathbb{1}_{A'})\Gamma_{AA'}}\right)\right]$$

$\hookrightarrow \underline{\rho_A^T \otimes \mathbb{1}_{A'}} |\Gamma\rangle\langle\Gamma|_{AA'}$
$= (\mathbb{1}_A \otimes \rho_{A'})|\Gamma\rangle\langle\Gamma|_{AA'}$

$$= \mathrm{Tr}_A\left[\mathcal{N}_{A'\to B}((\mathbb{1}_A \otimes \rho_{A'})\Gamma_{AA'})\right] \quad \left(\mathrm{Tr}_A[\Gamma_{AA'}] = \mathbb{1}_{A'}\right)$$
$$= \mathcal{N}_{A'\to B}(\rho_{A'}) \checkmark$$

$\hookrightarrow$ For any square matrix M,
$(M^T \otimes \mathbb{1})|\Gamma\rangle = (\mathbb{1} \otimes M)|\Gamma\rangle$
(Exercise!)

$$\Rightarrow \boxed{\mathcal{N}_{A\to B}(\rho_A) = \mathrm{Tr}_A\left[(\rho_A^T \otimes \mathbb{1}_B)\Gamma^{\mathcal{N}}_{AB}\right]}$$
$$= d_A \mathrm{Tr}_A\left[(\rho_A^T \otimes \mathbb{1}_B)\rho^{\mathcal{N}}_{AB}\right]$$

Examples of Q. Channels

1. **Depolarizing Channel**: $\mathcal{D}_p(\rho) = (1-p)\rho + \frac{p}{3}X\rho X + \frac{p}{3}Y\rho Y + \frac{p}{3}Z\rho Z \quad (0 \le p \le 1)$

$\left(\sigma_z\sigma_y = -i\sigma_x \Rightarrow \sigma_y = i\sigma_z\sigma_x\right)$.

$$= \left(1-\frac{4p}{3}\right)\rho + \frac{4p}{3}\mathrm{Tr}[\rho]\frac{\mathbb{1}}{2} \qquad \text{(for a qubit).}$$

$X \equiv \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ - bit flip

$Y \equiv \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ - bit and phase flip.

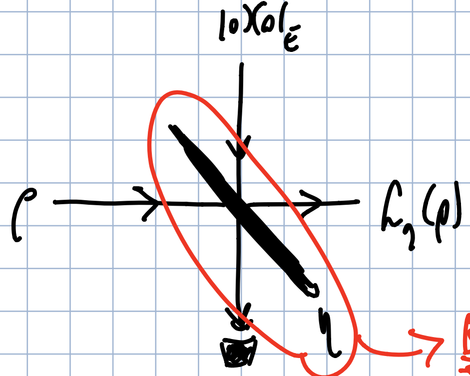
$Z \equiv \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ - phase flip

(*) $\frac{1}{4}\rho + \frac{1}{4}\sigma_x\rho\sigma_x + \frac{1}{4}\sigma_y\rho\sigma_y + \frac{1}{4}\sigma_z\rho\sigma_z = \mathrm{Tr}[\rho]\frac{\mathbb{1}}{2}$ (Exercise!)

Hint: Any operator ρ can be written as

$$\rho = \alpha_0\frac{\mathbb{1}}{2} + \alpha_1\sigma_x + \alpha_2\sigma_y + \alpha_3\sigma_z \Rightarrow \mathrm{Tr}[\rho] = \alpha_0$$

2. **Pure-loss Channel.** (model for optical channel)



→ Beamsplitter., $0 \le \eta \le 1$. Prob. η, photon gets through, prob. $1-\eta$ goes to environment.

Typically, $\eta = e^{-\alpha l}$, l: length of travel.

$\alpha \approx \frac{1}{22\,km}$ (depends on material).

((*) This is not a qubit channel, but a "bosonic" channel)

(*) **Special Case**: "Dual-rail" encoding of a qubit:

$|0\rangle \equiv |0,1\rangle \equiv |H\rangle$
$|1\rangle \equiv |1,0\rangle \equiv |V\rangle$
} Horizontal and vertical polarization modes.

Then, any qubit state $\boxed{\rho \mapsto (\mathcal{L}_\eta \otimes \mathcal{L}_\eta)(\rho) = \eta\,\rho + (1-\eta)|0,0\rangle\langle 0,0|}$

↳ "Erasure Channel"

3. Measurement Channel:

POVM $\{M_x\}_{x\in\mathcal{X}} \Rightarrow \mathcal{M}(\rho) = \sum_{x\in\mathcal{X}} \underbrace{\mathrm{Tr}[\rho M_x]}_{\text{outcome probabilities}} \underbrace{|x\rangle\langle x|_X}$ ← classical register storing the outcome. (like the probe) (measuring the probe gives the correct outcome probabilities).

$\left(\begin{array}{l} M_x \geq 0 \;\forall x\in\mathcal{X} \\ \sum_{x\in\mathcal{X}} M_x = \mathbb{1} \end{array} \right)$

If we measure A part of ρ_{AB}, then

$$\boxed{\mathcal{M}(\rho_{AB}) = \sum_{x\in\mathcal{X}} |x\rangle\langle x|_X \otimes \mathrm{Tr}_A[\rho_{AB}(M_x\otimes \mathbb{1}_B)] = \sum_{x\in\mathcal{X}} \underbrace{p(x)}_{\text{outcome prob.}} \underbrace{|x\rangle\langle x|_X}_{\text{classical register (stores the outcome)}} \otimes \underbrace{\rho_B^x}_{\text{post-measurement state of un-measured system}} = \rho_{XB} \;(\text{“classical-quantum” state}).}$$

$$\left(\rho_B^x = \frac{1}{p(x)} \mathrm{Tr}_A[(M_x\otimes \mathbb{1}_B)\rho_{AB}] \right)$$

(*) If Alice looks at the outcome (i.e., measures the probe) and sees the outcome x, then Bob's state is ρ_B^x. (She can then tell Bob what the outcome is so that he knows what state he has.).

(*) If Alice does not look at the outcome (i.e., ignores it or does not have access to it), then mathematically we describe this by tracing out X (partial trace is a quantum channel)

So the state of B has to be a random mixture of the possible states — (classical) uncertainty about the outcome

$$\Rightarrow \rho_B = \mathrm{Tr}_X(\rho_{XB}) = \sum_{x\in\mathcal{X}} p(x) \underbrace{\mathrm{Tr}_X[|x\rangle\langle x|]}_{1} \rho_B^x = \sum_{x\in\mathcal{X}} p(x)\rho_B^x$$

4. Quantum Instrument :

For POVM $\{M_x\}_{x\in\mathcal{X}}$, for outcome $x\in\mathcal{X}$, post-meas. state is

$$\rho^x = \frac{\sqrt{M_x}\,\rho\sqrt{M_x}}{\mathrm{Tr}[M_x\rho]}$$

More generally, $\rho \mapsto \rho^x = \mathcal{M}_x(\rho)$, where $\{\mathcal{M}_x\}_x$ is a set of CP maps such that $\sum_{x\in\mathcal{X}} \mathcal{M}_x$ is TP. $\{\mathcal{M}_x\}_x$ is called a q. instrument.

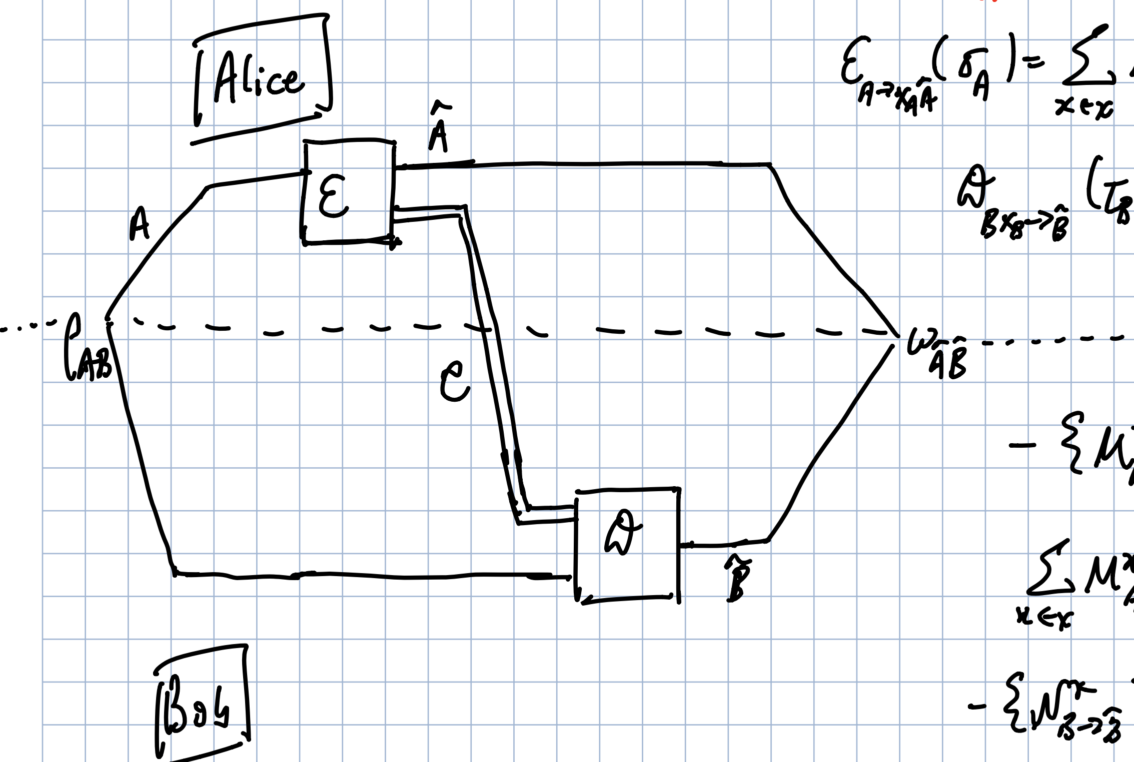
$$\mathcal{E}(\rho) = \sum_{x\in\mathcal{X}} \mathcal{M}_x(\rho)\otimes |x\rangle\langle x|_X \qquad \text{(Quantum instrument channel)}$$

classical register, holds the meas. outcome.

5. LOCC Channel : $\mathcal{L}^{\rightarrow}_{AB\to\hat{A}\hat{B}} = \mathcal{D}_{BX_B\to\hat{B}} \circ \underbrace{\mathcal{C}_{X_A\to X_B}}_{} \circ \mathcal{E}_{A\to X_A\hat{A}}$

(This is a one-way LOCC channel from Alice to Bob)

Classical comm. channel (noiseless).



$$\mathcal{E}_{A\to X_A\hat{A}}(\sigma_A) = \sum_{x\in\mathcal{X}} \mathcal{M}^x_{A\to\hat{A}}(\sigma_A)\otimes|x\rangle\langle x|_{X_A}$$

$$\mathcal{D}_{BX_B\to\hat{B}}(\tau_B\otimes|x\rangle\langle x|_{X_B}) = \mathcal{N}^x_{B\to\hat{B}}(\tau_B)$$

- $\{\mathcal{M}^x_{A\to\hat{A}}\}_{x\in\mathcal{X}}$ CP such that $\sum_{x\in\mathcal{X}}\mathcal{M}^x_{A\to\hat{A}}$ TP (i.e., q. instrument).

- $\{\mathcal{N}^x_{B\to\hat{B}}\}_{x\in\mathcal{X}}$ CPTP maps.

A general two-way LOCC channel is a concatenation of one-way LOCC channels